Combine two algebraic System into a Single System to get the Vector space

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ABSTRACT

In this paper we consider the matter of combining two different mathematical system (algebraic system) into a single system Known as a Vector Space .
IF we combine two System $(V,+)$ *and* $(F,+,.)$
in one system ,then we obtain to Vector Space
$f:V\times V\to V$ *is called internal operation* $((addition))$.
The operation F;$F\times V\to V$ *is called external binary operation* $(scalar\ multiplication)$.
Then we give some definitions and examples to illustrate the main results.

Keywords: Vector Space V(F) –The Field(F) – Subspaces **W - linear dependent(LD)- Linear independent (LI).**

INTRODUCTION :

Before we give the formal definition of a vector space ,we need to define of the field

Definition: A field is a Set F of numbers with property that if $a,b\in F$ **,then (**$a+b,a-b,ab,{}^{a}/_{b}$ *are also in* $F,b\neq 0)$.

The ((field means real numbers R, rationl numbers Q,Complex numbers C)).

We give the definition of a vector space over the a field . It will be clear that, under the definitions of two system addition and scalar multiplication over F, and we study it's properties ,then we give definition of Subspace with exampes and some theorems.

Lastly we come the most important concepts in the theory of vector space, (linear indpenden and dependent) by giving their definitions and some examples

(Vector Space)

Definition(1) :

The system (V,⊕) , (F,+, .) is called a Vector Space if and only if :

a- (F,+,.) is a field with Identity 0 in addition and 1 which Identity in multiplication.

b- (V,⊕) is a commutative group, whose elements are called vectors

1-$\forall\alpha,\beta\ V,\ \alpha\oplus+\beta\ V$ $(closure\ property)$.

$2-\ \oplus$ *addition is commutative* $\alpha\oplus\beta=\beta\oplus\alpha$.

3– ⊕ **addition is associative** $(\alpha\oplus\beta)\bigoplus\gamma=\alpha\oplus\left(\beta\bigoplus\gamma\right)$.

4 –***Zero vector*** $\alpha\bigoplus 0=\alpha,\quad\forall\alpha\in V$.

5– **The inverse such that** $\alpha + (-\alpha) = 0$.

$c - Multiplication$

1- For each $a , b \in F$ and α , β V ;

$(a + b) \odot \alpha = (a \odot \alpha) \oplus (b \odot \alpha)$.

2- $(a . b) \odot \alpha = a \odot (b \odot \alpha)$

3-The Identity is 1 such that $1 \odot \alpha = \alpha$.

Example (1-1):

Let (R, + , .) be a ring of real numbers and (C, + , .) be comlex numbers then the field

(C,+, .) a Vector Space of real numbers.

Example(1 -2):

IF (F,+ .) be a field and V be the set of all m x n Matrices over the field F.

Solution:

IF $A = [a_{ij}] \in F$ $where$ $i = 1, 2, \dots, m$ $(rows)$

and $j = 1, 2, 3, \dots, n$ $(colmuns)$

If $B = [b_{ij}] \in F$,

The sum of two Matrices is

$A + B = [a_{ij} + b_{ij}]$.

The product of scalar $k \in F$

$k A = [ka_{ij}]$ is $called$ $scalar$ $multiplication$.

Now , we want to prove V(F) is a Vector Space.

Let $A , B \in V$ $, and$ $T_1 , T_2 \in \mathrm{F}$, $\mathbf{then}$

1- $(T_1 + T_2) A = T_1 A + T_2 A$.

2- $T_1 (A + B) = T_1 A + T_1 B$.

3- $(T_1 T_2) A = T_1 (T_2 A)$.

4- The $Identity$ $I . A = A$.

Thus V(F) is a Vector space.

Example (1-3):

1 . The set $V = F^n$ of an $n - tuples$ $(x_1, \dots, x_n)^T$ is a $Vector$ $space$ $sine$

$$(x_1, \dots, x_n)^T + (y_1, \dots, y_n)^T = (x_1 + y_1, \dots, x_n + y_n)^T$$

$$\alpha . (x_1, \dots, x_n)^T = (\alpha x_1, \dots, \alpha x_n).$$

$Then$ $V(F)$ is a $Vector$ $Space$ $over$ the $field$ $(F, +, .)$.

Theorem 1-1 :

IF V(F) is a vector Space and $x , y \in V$ and $c \in F$ $, then$

1- $0x = \bar{0}$,

2- $c \bar{0} = \bar{0}$,

3 – $(cx) = c (-x)$,

4- $a (x - y) = ax - ay$.

Proof:

We know $0 + 1 = 1$. $Then$ $x (0 + 1) = 0x + 1x = \bar{0} + x$

$0x + x = \bar{0} + x$ by $cancellation$ law $gives$ $0x = \bar{0}$.

next we want to prove $c\ \overline{0} = \overline{0}$.

$we\ \ have\ \ c\ \overline{0} + cx = c\left(\overline{0} + x\right) = \overline{0} + cx$. $by\ \ concellation\ \ law$ $we\ \ get\ \ the\ \ result\ \ c\ \overline{0} = \overline{0}$.

We observe that $\overline{0} = 0x\ by\ \ first\ \ prove$

this mean that $(-c)\ x = -(cx)$. $Similarly$

$\overline{0} = c\ \overline{0} = c\ (x) + c\ (-x)$ $which\ \ implies$

$c\ (-x) = -(cx)$.

Lastely Prove (4)

$a\ (x-y) = a\ [x + (-y)] = ax + a\ (-y)$

$= ax - (ay)$

finally we get the results.

<u>SUBSPACES(2)</u>

<u>Definition: (2-1)</u>

Let V be a Vector Space over the field (F,+.). $If\ \ W \neq \emptyset\ \ a\ subset\ \ of\ Vis\ \ called$ **a subspace of V if and only if W itself is a Vector space over F under the operation of V.**

From the definition (W,+) is a subgroupe of (V,+).

W is closed under difference $,\alpha\ ,\beta\ \ W\ ,then\ \ \alpha-\beta\ \ W$.

Also W is closed under Scalar multiplication.

<u>Example : (2-1)</u>

Consider the set W of vector space in $V_3(F)$ $whose\ \ components\ \ add\ \ up\ \ to$ $Zero$;$W = \{(x_1, x_2, x_3 : x_1 + x_2 + x_3 = 0)\}$

$If\ \ (x_1, x_2, x_3), (y_1, y_2, y_3) \in W$, $then$

$we\ \ check\ \ their\ \ differencee$

$(x_1, x_2, x_3) - (y_1, y_2, y_3) = (x_1 - y_1, x_2 - y_2, x_3 - y_3)$

and

$(x_1 + x_2 + x_3) - (y_1 + y_2 + y_3) = 0$

$\therefore Wis\ \ Subgroupe\ \ under\ \ addition$

$\forall c \in F\ ,c\ (x_1, x_2, x_3) = (c\ x_1, cx_2, c\ x_3)$

$= c\ (x_1 + x_2 + x_3) = (0) = 0$.

$\therefore W\ is\ \ Subspace\ \ of\ \ V_3(F)$.

$Which\ \ shows\ \ W\ is\ closed\ \ under\ \ scalar\ \ multiplication$.

Example (2-2):

Consider the set W is all elements from the space $M_2(F)$ **if W=**$\left\{\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}, \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\right\}$

,then

We want to prove this system is a Subspace

$\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix} - \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} = \begin{bmatrix}-2 & 0\\ 0 & 2\end{bmatrix} \in W$,

$k\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix} = \begin{bmatrix}-k1 & 0\\ 0 & k\end{bmatrix} \in W$,

$\therefore W\ (F)\ is\ \ subspace\ \ of\ \ the\ \ vector\ \ space\ \ M_2$

<u>Theorem (2-1):</u>

Let V be a vector space over the field (F,+,.) . IF W$\subseteq V$, $W \neq \emptyset$.
$Then\ W(F)\ is\ a\ subspace\ of\ V(F)\ if\ u,v \in W and\ \alpha,\beta\ F \Rightarrow \alpha u + \beta \in W$.

Example (3-2):
Let W=$\{(a,0,0); a \in R\}\ is\ a\ subspace\ of\ R^3,\ for\ let\ u\ (a,0,0), v=(b,0,0)\ a$
$\alpha,\beta \in R$.
$Then\ \alpha u + \beta v = \alpha(a,0,0) + \beta(b,0,0) = (\alpha a + \beta b, 0, 0) \in W$.
$Hence\ W\ is\ a\ subspace\ of\ R^3$.

Theorem (2 -2):
IF $W_1(F)\ and\ W_2(F)\ are\ subspace\ of\ V(F), then\ (W_1 \cap W_2)(F)$
$is\ so\ subspace\ of\ V(F)$.
Proof:
The Set $W_1 \cap W_2\ is\ not\ empty\ , for\ \bar{0} \in W_1 \cap W_2$.
Now let $u+v \in W_1,\ c\ \alpha\ W_2$,
and $u+v \in W_2,\ c\ \alpha\ W_2$,
implies $u+v \in W_1 \cap W_2$
and $c\ \alpha \in W_1 \cap W_2$.
This Prove the theorem.
Definition (2-2):
Let A and B be subspace of a vector space V.Then V is called direct sum of A and B if and only if:

i- A+ B=V,

ii- $A \cap B = \{0\}$,

simbolly the direct sum is v =$A \bigoplus B$.

Example (4- 2):
Let $A = \{(a,b,0): a,b \in R\}\ and\ B = \{(0,0,c): c \in R\}\ in\ V_3(R)$
Clearly A $\cap B = 0$,
A+B=(a,b,0) + (0,0,c) = (a,b,c)$\in V_3(R)$,
hence $V_3(R) = A \oplus$ B.

<u>Theorem(3-2)</u>:
If $W_1(F)\ and\ W_2(F)$ **are subspace of the vector space V(F), then their union**$(W_1 \cup W_2)$ **is not necessarily a vector space V(F).**
Proof:
We Know by the definition of a subspace ,if $(W_1 \cup W_2, +)\ is\ not\ a\ subspace$
$of\ (V,+), then\ (W_2 \cup W_2)(F)\ is\ not\ a\ subspace$.
Example:(4)

Let $A = \{(a, 0, 0) : a \in R\}, B = \{(0, b, 0) : b \in R\}$.

Clearly A and B are subspace of R^3.

But $A \cup B$ *is not a subspace of* R^3.

Since $(a, 0, 0) + (o, b, 0) = (a, b, 0) \notin A \cup B$.

<u>3-LINEAR DEPENDENCE AND LINEAR INDEPENDECE</u>:

We come to one of most useful concepts in vector space, that of linear dependence and linear independence

DEFINITIONS: (3-1)

1- Let $S = \{v_1, \dots, v_k\} \subset V$ *, a vector space . We say that S is* linearly dependent (LD) if there are scalars $a_1, \dots\dots, a_k$ *not all zero or ((an infin* *which*

   $a_1 v_1 + a_2 v_2 + \dots + a_k v_k = 0$. *

2- Let $S = \{v_1, \dots, v_k\} \subset V$ *, a vector space . We say that S is* linearly independent (exactly one solution) if there are scalars $a_1, \dots\dots, a_k$ *which*

   $a_1 v_1 + a_2 v_2 + \dots + a_k v_k = 0$.

   $a_1 = a_2 = \dots = a_k = 0$ *

Example (3-1)

Let $V = R^3$ *and* $S = \{v_1, v_2, v_3\}$ *the vectors defined as*

$v_1 = (1, 0, 0)$

$v_2 = (0, 1, 0)$

$v_3 = (0, 0, 1)$

are linear independent , since $a_1 v_1 + a_2 v_2 + \dots + a_k v_k = 0$.

$a_1 (1, 0, 0) + a_2 (0, 1, 0) + a_3 (0, 0, 1) = 0$

$= (a_1, a_2, a_3) = (0, 0, 0)$

implies

$a_1 = a_2 = a_3 = 0$.

Example(3-2):

Show that the set $\{v_1, v_2, v_3\}$ *is linearly dependent if the set* $\{v_1 + a\ v_2 + b\ v_3, v_2, v_3\}$ is linearly dependent ,where $v_1, v_2, v_3 \in$ *vector space* ,a,b belong to the field .

Solution:

we use the equation $a_1 v_1 + a_2 v_2 + \dots + a_k v_k = 0$.

$a_1 (v_1 + a\ v_2 + b\ v_3) + a_2 (v_2) + a_3 (v_3) = 0$ ....(1)

if $a_1 \neq 0$ *, then* v_1, v_2, v_3 *are linearly dependent*

if $a_1 = 0$, *the equation* (1) *becomes* $a_2 v_2 + a_3 v_3 = 0$ *implies since* $\{v_1, v_2, v_3\}$ *is linearly dependent* .

Example (3-3):

Show that the vectors $V_1 = (1, 2, 3), V_2 = (3, -2, 0)$ are **linearly independent.**

Solution:

To show that we use the formula

$$a_1 v_1 + a_2 v_2 = 0 \ ,$$

$$which \quad means \qquad a_1 + 3a_2 = 0$$

$$2a_1 - 2a_2 = 0$$

$$3a_2 = 0 \ .$$

Implies $a_1 = a_2 = 0$.

Corollary :

If $0 \in S = \{v_1, \dots, v_k\}$, $then \quad S \ is \quad lineary \quad dependent$.

Conclsion :

In fact the groups needs one binary operation and the mathematical systems like rings ,fields needs two operations but in vector space we combine two algebraic system .This system (Vector Space) is very useful in our education like Solving the System of linear equations or we can find the solutions by change the equations to Matrices .Solving system of linear equations be more easily viewed from the perspective of a vector space.

The vector space to mean any type of mathematical object that can multiplied by numbers and add together . This way, the theorem start with the Phrase The Vector Space.

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