Second Hankel Determinant for Certain Class of Functions Defined by Differential Operator

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Abstract:

The objective of this paper is to obtain an upper bound of second Hankel determinant for a class of functions $M^k_{\alpha,\beta,\lambda,\delta}(\varphi)$ defined in the open unit disc U by using the differential operator $D^k_{\alpha,\beta,\lambda,\delta}f(z)$, also, we give particular values to the parameters A, B and k to study special cases of the results of this article. The class $M^k_{\alpha,\beta,\lambda,\delta}(\varphi)$ and the differential operator $D^k_{\alpha,\beta,\lambda,\delta}f(z)$ were defined by S. F. Ramadan and M. Darus [8].

Key words: differential operator, Second Hankel determinant, Starlike functions, Subordination property.

1.Introduction:

In 1976, Noonan and Thomas [4] defined the q^{ht} Hankel determinant of $f(z)$ for $q, n \in N$ as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} \cdots & a_{n+q-1} \\ a_{n+1} & \cdots & \cdots \\ \vdots & \vdots & \vdots \\ a_{n+q-1} & \cdots & a_{n+2q-2} \end{vmatrix}$$

That is, for the complex sequence , $a_n, a_{n+1}, a_{n+2}, \ldots$ the Hankel matrix is the infinite matrix whose $(i,j)^{th}$ entry a_{ij} is defined by

$$a_{ij} = a_{n+i+j-2}, \qquad (i, j, n \in N).$$

This determinant was discussed by several authors particularly for the cases when $q = 2$, $n = 1$, $a_1 = 1$ and $q = 2$, $n = 2$, that is

$$H_2(1) = \left| a_3 - a_2^2 \right| \quad \text{and} \quad H_2(2) = \left| a_2 a_4 - a_3^2 \right|$$

where $H_2(1)$ is known as the Fekete-Szeqö problem and $H_2(2)$ refer to the second Hankel determinant. For example, (Janteng, Abdul Halim, and Darus [1]), (Gagandeep Singh, Gurcharanjit Singh[2]) and (Panigrahi, G. Murugusundaramoorthy [9]), and many others have obtained sharp upper bounds of $H_2(2)$ for different classes of analytic functions.

In the present paper, and by making use of the differential operator $D^k_{\alpha,\beta,\lambda,\delta}f(z)$ we will obtain the upper bound of the second Hankel determinant for the class $M^k_{\alpha,\beta,\lambda,\delta}(\phi)$ that well be defined below .

Let A be the class of analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

defined in the open unit disk $U=\{z\in\mathbb{C}:|z|<1\}$.

S. F. Ramadan and M. Darus [8] defined the following differential operator For a function $f\in A$ as follows:

$$D^{k}_{\alpha,\beta,\lambda,\delta}f(z)=z+\sum_{n=2}^{\infty}[(\lambda-\delta)(\beta-\alpha)(n-1)+1]^{k}a_{n}z^{n}, \qquad (2)$$

Where $\alpha,\beta,\lambda,\delta\geq 0$, $\lambda>\delta$, $\beta>\alpha$ and $k\in\{0,1,2,...\}$.

Also in the same paper, they defined the class $M^{k}_{\alpha,\beta,\lambda,\delta}(\phi)$ as follows

Definition 1.[8] Let $\phi(z)$ be a univalent starlike function with respect to 1 which maps the unit disk U onto a region in the right half plane which is symmetric with respect to the real axis, $\phi(0)=1$ and $\varphi'(0)>0$.

A function $f\in A$ is in the class $M^{k}_{\alpha,\beta,\lambda,\delta}(\phi)$ if

$$\frac{z\left(D^{k}_{\alpha,\beta,\lambda,\delta}f(z)\right)'}{D^{k}_{\alpha,\beta,\lambda,\delta}f(z)}\prec\varphi(z). \qquad (3)$$

If J is the class of functions $G(z)$ defined as

$$G(z)=(1-(\lambda-\delta)(\beta-\alpha))f(z)+(\lambda-\delta)(\beta-\alpha)zf'(z)$$

$$=z+\sum_{n=2}^{\infty}[(\lambda-\delta)(\beta-\alpha)(n-1)+1]^{k}a_{n}z^{n},$$

For $\alpha,\beta,\lambda,\delta\geq 0$, $\lambda>\delta$, $\beta>\alpha$ and $k\in\{0,1,2,...\}$.

Then $G(z)$ can be considered as the analytic function in U.

Also, let S be defined as the class of functions $G(z)\in J$, which is univalent in U.

Using Schwarzian functions which are analytic in U and satisfting

$$w(z)=\sum_{n=1}^{\infty}d_{n}z^{n},$$

the conditions $w(0)=0$ and $|w(z)|<1$. let f and g be two analytic functions in U. Then, f is a subordinate to g ($f\prec g$) if $f(z)=g(w(z))$ is satisfied.

Now, if we set $\phi(z)=\frac{1+Az}{1+Bz}=1+(A-B)z-B(A-B)z^{2}+B^{2}(A-B)z^{3}+\ldots,$

$(-1\leq B<A\leq 1)$ in (3) we can write that

$$\frac{z\left(D^{k}_{\alpha,\beta,\lambda,\delta}f(z)\right)'}{D^{k}_{\alpha,\beta,\lambda,\delta}f(z)}\prec\frac{1+Az}{1+Bz}, \qquad (4)$$

And we can write the class $M^{k}_{\alpha,\beta,\lambda,\delta}(\phi)$ as $M^{k}_{\alpha,\beta,\lambda,\delta}(A,B)$

let $M^{k}_{\alpha,\beta,\lambda,\delta}(A,B)$ be a subclass of the functions $G(z)\in J$ and satisfy the condition

$$\frac{zG'(z)}{G(z)}\prec\frac{1+Az}{1+Bz}, \qquad (-1\leq B<A\leq 1). \qquad (5)$$

Where $M^{k}_{\alpha,\beta,\lambda,\delta}(A,B)$ is subclass of starlike functions and $M^{0}_{\alpha,\beta,\lambda,\delta}(A,B)\equiv S^{*}(A,B)$ and $M^{0}_{\alpha,\beta,\lambda,\delta}(1,-1)\equiv S^{*}$.

The class S^{*} is the class of starlike functions and studied by Goel and Mehrok [7].

2. Preliminary Results

Let P denote the class of functions p analytic in U, for which Re{$p(z)$} > 0,

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \ldots = \left[1 + \sum_{n=1}^{\infty} p_n z^n\right], \qquad z \in U. \tag{6}$$

In order to investigate our mean result, we need the following lemmas:

Lemma 2.1. [5] If $p \in P$, then $|p_k| \le 2$, $\quad (k = 1, 2, 3, \ldots)$.

Lemma 2.2. [3] , [6] If $p \in P$, then

$$2p_2 = p_1^2 + (4 - p_1^2)x,$$

$$4p_3 = p_1^3 + 2p_1(4 - p_1^2)x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)z,$$

For some x and z satisfying $|x| \le 1$, $|z| \le 1$ and $p_1 \in [0, 2]$.

3. Main Results

Theorem 3.1: If $G(z) \in M_{\alpha,\beta,\lambda,\delta}^{k}(A, B)$, then

$$|a_2 a_4 - a_3^2| \le \frac{(A - B)^2}{4[2(\lambda - \delta)(\beta - \alpha) + 1]^{2k}}, \tag{7}$$

Proof. If $G(z) \in M_{\alpha,\beta,\lambda,\delta}^{k}(A, B)$, then there exists a Schwarz function $w(z)$ such that

$$\frac{zG'(z)}{G(z)} = \varphi(w(z)), \qquad (z \in U). \tag{8}$$

Where

$$\varphi(z) = \frac{1 + Az}{1 + Bz} = 1 + (A - B)z - B(A - B)z^2 + B^2(A - B)z^3 + \ldots$$

$$= 1 + E_1 z + E_2 z^2 + E_3 z^3 + \ldots \tag{9}$$

Furthermore, the function $p_1(z)$ can be defined as follows:

$$p_1(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + b_1 z + b_2 z^2 + b_3 z^3 + \ldots \tag{10}$$

Now, we can write $R(p_1(z))>0$ and $p_1(0) = 1$.

After that, we define the function $h(z)$ by

$$h(z) = \frac{zG'(z)}{G(z)} = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \ldots. \tag{11}$$

From the equations (8), (10) and (11) we have

$$h(z) = \varphi\left(\frac{p(z) - 1}{p_1(z) + 1}\right) = \varphi\left(\frac{b_1 z + b_2 z^2 + b_3 z^3 + \ldots}{2 + b_1 z + b_2 z^2 + b_3 z^3 + \ldots}\right)$$

$$= \varphi\left[\frac{1}{2} b_1 z + \frac{1}{2}\left(b_2 - \frac{b_1^2}{2}\right) z^2 + \frac{1}{2}\left(b_3 - b_1 b_2 - \frac{b_1^3}{4}\right) z^3 + \ldots\right]$$

$$= 1 + \frac{E_1 b_1}{2} z + \left[\frac{E_1}{2}\left(b_2 - \frac{b_1^2}{2}\right) + \frac{E_2 b_1^2}{4}\right] z^2 + \left[\frac{E_1}{2}\left(b_3 - b_1 b_2 + \frac{b_1^3}{4}\right) + \quad \frac{E_2 b_1}{2}\left(b\right.\right.$$

Thus,

$$c_1 = \frac{E_1 b_1}{2}, \qquad c_2 = \frac{E_1}{2}\left(b_2 - \frac{b_1^2}{2}\right) + \frac{E_2 b_1^2}{4},$$

and

$$c_3 = \frac{E_1}{2}\left(b_3 - b_1 b_2 + \frac{b_1^3}{4}\right) + \frac{E_2 b_1}{2}\left(b_2 - \frac{b_1^2}{2}\right) + \frac{E_3 b_1^3}{8}. \qquad (12)$$

Now, by employing (9) and (11) in (12) we get

$$[(\lambda - \delta)(\beta - \alpha) + 1]^k \, a_2 = c_1$$

Then,

$$a_2 = \frac{c_1}{[(\lambda - \delta)(\beta - \alpha) + 1]^k} = \frac{E_1 b_1}{2\,[(\lambda - \delta)(\beta - \alpha) + 1]^k} = \frac{(A - B)\, b_1}{2\,[(\lambda - \delta)(\beta - \alpha) + 1]^k}. \qquad (1$$

After that,

$$2\,[2(\lambda - \delta)(\beta - \alpha) + 1]^k \, a_3 = c_2 + c_1^2$$

Then,

$$a_3 = \frac{1}{2\,[2(\lambda - \delta)(\beta - \alpha) + 1]^k}\,[c_2 + c_1^2]$$

$$= \frac{1}{2\,[2(\lambda - \delta)(\beta - \alpha) + 1]^k}\left[\frac{E_1}{2}\left(b_2 - \frac{b_1^2}{2}\right) + \frac{E_2 b_1^2}{4} + \frac{E_1^2 b_1^2}{4}\right]$$

$$= \frac{1}{2\,[2(\lambda - \delta)(\beta - \alpha) + 1]^k}\left[\frac{b_2 E_1}{2} - \frac{E_1 b_1^2}{4} + \frac{E_2 b_1^2}{4} + \frac{E_1^2 b_1^2}{4}\right]$$

$$= \frac{1}{2\,[2(\lambda - \delta)(\beta - \alpha) + 1]^k}\left[\frac{(A - B)\, b_2}{2} - \frac{(A - B)\, b_1^2}{4} - \frac{B\,(A - B)\, b_1^2}{4} + \frac{(A - B)^2 b_1^2}{4}\right]$$

$$= \frac{A - B}{8\,[2(\lambda - \delta)(\beta - \alpha) + 1]^k}\left[2 b_2 - b_1^2 - B\, b_1^2 + (A - B)\, b_1^2\right]$$

$$= \frac{A - B}{8\,[2(\lambda - \delta)(\beta - \alpha) + 1]^k}\left[2 b_2 - b_1^2 - 2B\, b_1^2 + A b_1^2\right]$$

$$= \frac{A - B}{8\,[2(\lambda - \delta)(\beta - \alpha) + 1]^k}\left[2 b_2 + (A - 2B - 1) b_1^2\right]. \qquad (14)$$

And

$$4\,[3(\lambda - \delta)(\beta - \alpha) + 1]^k \, a_4 = c_3 + c_2\,[(\lambda - \delta)(\beta - \alpha) + 1]^k \, a_2$$

$$+c_1[2(\lambda-\delta)(\beta-\alpha)+1]^k a_3+[3(\lambda-\delta)(\beta-\alpha)+1]^k a_4,$$

$$3[3(\lambda-\delta)(\beta-\alpha)+1]^k a_4=c_3+c_2[(\lambda-\delta)(\beta-\alpha)+1]^k a_2 + c_1[2(\lambda-\delta)(\beta-\alpha)+1]^k a_3$$

$$=c_3+c_2c_1+c_1[2(\lambda-\delta)(\beta-\alpha)+1]^k a_3,$$

$$=c_3+\frac{3c_1c_2}{2}+\frac{c_1^3}{2}$$

,

$$a_4=\frac{c_3}{3[3(\lambda-\delta)(\beta-\alpha)+1]^k}+\frac{c_1c_2}{2[3(\lambda-\delta)(\beta-\alpha)+1]^k}+\frac{c_1^3}{6[3(\lambda-\delta)(\beta-\alpha)+1]^k}$$

$$=\frac{1}{6[3(\lambda-\delta)(\beta-\alpha)+1]^k}\left[2c_3+3c_1c_2+c_1^3\right]$$

so that,

$$=\frac{1}{6[3(\lambda-\delta)(\beta-\alpha)+1]^k}\left[E_1\left(b_3-b_1b_2+\frac{b_1^3}{4}\right)+E_2b_1\left(b_2-\frac{b_1^2}{2}\right)+\frac{E_3b_1^3}{4}+\frac{3E_1b_1}{2}\left(\frac{E_1}{2}\left(b_2-\frac{b_1^2}{2}\right)+\frac{E_2b_1^2}{4}\right)+\frac{E_1^3b}{8}\right.$$

$$=\frac{(A-B)\left[8b_3+(6A-14B-8)b_1b_2+(A^2+6B^2-5AB-3A+7B+2)\,b_1^3\right]}{48[3(\lambda-\delta)(\beta-\alpha)+1]^k}. \qquad (15)$$

From (13), (14) and (15) we find that

$$a_2a_4-a_3^2=\frac{(A-B)^2}{192}$$

$$\times\left[\frac{R(b_1,b_2,b_3,A,B,\lambda,\delta,\beta,\alpha)}{[(\lambda-\delta)(\beta-\alpha)+1]^k[2(\lambda-\delta)(\beta-\alpha)+1]^{2k}[3(\lambda-\delta)(\beta-\alpha)+1]^k}\right]. \qquad (16)$$

Where

$$R(b_1,b_2,b_3,A,B,\lambda,\delta,\beta,\alpha)=16[2(\lambda-\delta)(\beta-\alpha)+1]^{2k}b_1b_3-12[(\lambda-\delta)(\beta-\alpha)$$

If limm2.1and lemma 2.2 are applied to (16) we have

$$\left|a_2a_4-a_3^2\right|=\frac{(A-B)^2\left|T(A,B,\lambda,\delta,\beta,\alpha,b_1,x,z)\right|}{192[(\lambda-\delta)(\beta-\alpha)+1]^k[2(\lambda-\delta)(\beta-\alpha)+1]^{2k}[3(\lambda-\delta)(\beta-\alpha)+1]^k}$$

Such that

$$T(A,B,\lambda,\delta,\beta,\alpha,b_1,x,z)=(2A^2[2(\lambda-\delta)(\beta-\alpha)+1]^{2k}-3A^2[(\lambda-\delta)(\beta-\alpha)-$$

By assuming that $b_1=b$ and $b\in[0,2]$, with triangular inequality and $|z|\le 1$ the following

$$\left|a_2a_4-a_3^2\right|\ge\frac{(A-B)^2F(\gamma)}{192[(\lambda-\delta)(\beta-\alpha)+1]^k[2(\lambda-\delta)(\beta-\alpha)+1]^{2k}[3(\lambda-\delta)(\beta-\alpha)+1]^k},$$

is also obtainable where $\gamma = |x| \le 1$ and

$$F(\gamma) = (|A|^2 |2[2(\lambda-\delta)(\beta-\alpha)+1]^{2k} - 3[(\lambda-\delta)(\beta-\alpha)+1]^k [3(\lambda-\delta)(\beta-\alpha)+1]^k| + 12|B|^2 |[2(\lambda-\delta)(\beta-\alpha)+1]^{2k} - [(\lambda-\delta)(\beta-\alpha)+1]^k [3(\lambda-\delta)(\beta-\alpha)+1]^k| + 2|AB| |6[(\lambda-\delta)(\beta-\alpha)+1]^k [3(\lambda-\delta)(\beta-\alpha)+1]^k - 5[(\lambda-\delta)(\beta-\alpha)+1]^{2k}|] b^4 + 8[2(\lambda-\delta)(\beta-\alpha)+1]^{2k} b(4-b^2) + [3(\lambda-\delta)(\beta-\alpha)+1]^k| + 2|B| |7[2(\lambda-\delta)(\beta-\alpha)+1]^{2k} - 6[(\lambda-\delta)(\beta-\alpha)+1]^k [3(\lambda-\delta)$$

As $F(\gamma)$ is an increasing function, $Max\ F(\gamma) = F(1)$ is also satisfactorily applicable.

Therefore, $$|a_2 a_4 - a_3^2| \ge \frac{(A-B)^2}{192[(\lambda-\delta)(\beta-\alpha)+1]^k [2(\lambda-\delta)(\beta-\alpha)+}$$

Where $g(b) = F(1)$.

So

$$g(b) = (|A|^2 |2[2(\lambda-\delta)(\beta-\alpha)+1]^{2k} - 3[(\lambda-\delta)(\beta-\alpha)+1]^k [3(\lambda-\delta)(\beta-\alpha)+1]^k| + 12 [24|A| |[(\lambda-\delta)(\beta-\alpha)+1]^{2k} - [(\lambda-\delta)(\beta-\alpha)+1]^k [3(\lambda-\delta)(\beta-\alpha)+1]^k| + 8|B| |7[2(\lambda$$

Now

$$g'(b) = 4[|A|^2 |2[2(\lambda-\delta)(\beta-\alpha)+1]^{2k} - 3[(\lambda-\delta)(\beta-\alpha)+1]^k [3(\lambda-\delta)(\beta-\alpha)+1]^k| + 2[24|A| |[(\lambda-\delta)(\beta-\alpha)+1]^{2k} - [(\lambda-\delta)(\beta-\alpha)+1]^k [3(\lambda-\delta)(\beta-\alpha)+1]^k| + 8|B| |7[2(\lambda$$

Since $g'(b) < 0$ for $b \in [0,2]$, the maximum value of $g(b)$ at $b=0$ indicates that $Max\ g(b) = g(0)$.

Thus, (7) can be achieved from (17), that is

$$\frac{(A-B)^2}{\beta-\alpha)+1]^k [2(\lambda-\delta)(\beta-\alpha)+1]^{2k} [3(\lambda-\delta)(\beta-\alpha)+1]^k} \times 48 [[(\lambda-\delta)(\beta-\alpha)+1]^k [3(\lambda-\delta)(\beta-\alpha)+1]^k]$$

Whereby for $b_1 = 0$, $b_2 = 2$, and $b_3 = 0$ the resulting value is sharp.

Based on Theorem 3.1, we shall obtain several corollaries given as follows:

Corollary 3.1. If $F(z) \in S^*$, then

$$|a_2 a_4 - a_3^2| \le \frac{1}{[2(\lambda-\delta)(\beta-\alpha)+1]^{2k}}$$

This is obtained for A =1 and B = -1.

By applying $k = 0$ in Theorem3.1, the finding obtained is in line with Singh and Singh [2] stated below:

Corollary 3.2. If $f(z) \in S^*(A, B)$, then

$$|a_2 a_4 - a_3^2| \le \frac{(A-B)^2}{4}$$

For $A = 1$, $B = -1$ and $k = 0$, Theorem 3.1 gives the following result due to Janteng, Halim and Darus [1].

Corollary 3.3. If $f(z) \in S^*$, then $\left|a_2 a_4 - a_3^2\right| \le 1$.

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