Modification of the Tamimi- Ansari method to solve Sine- Gordon equation

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Abstract

In this paper, we aim to solve the Sine-Gordon equation by employing a modified version of the Tamimi-Ansari method. This particular method relies on the utilization of Taylor series and Chebyshev approximation polynomials. Additionally, we will examine the convergence of these cases by employing various theories, including the Banach fixed point theorem. To conduct the necessary calculations for this research, we will utilize Mathematica 13.2 software.

Keywords: Sine- Gordon equation-modified Tamimi- Ansari method- Taylor series- Chebychev approximation polynomials.

1. Introduction

The Sine-Gordon equation is a nonlinear partial differential equation that originated in the nineteenth century during the study of differential geometry problems. Its significance grew in the 1970s when it was discovered that the equation had implications for solution solutions [1]. The equation also appears in various other physical applications and relativistic field theory. It can be expressed as follows:

$$u_{tt} - u_{xx} + \sin(u) = 0 \qquad (1)$$

with the initial conditions:

$$u(x,0) = f(x), u_t(x,0) = g(x) \qquad (2)$$

where $u = u(x,t)$. Due to the difficulty of obtaining exact solutions for this equation, numerous numerical and analytical methods have been employed to obtain approximate solutions. For instance, Herbst et al. [2] utilized an explicit symplectic method to obtain numerical results for the Sine-Gordon equation. Other methods utilized include the Adomian Decomposition method [3, 4], B-Spline collection method [5], homotopy analysis method [6], variational iteration method [7, 8, 9, 10], Daftardar-Gejji and Jafari Method [11], among others.

In this study, we will introduce a modification of the Tamimi-Ansari method to approximate the solution of the Sine-Gordon equation with given initial conditions. Specifically, we will employ the Taylor series [10] and Chebyshev approximation [12] polynomials in two separate cases to approximate the nonlinear function sin(u) in the correction functional. Additionally, we will discuss the convergence between these cases employing the fixed point theorem.

2. Basics of the TAM method

In this section, we will explain the idea of TAM method [13]

Suppose that we have the general differential equation

$$L\big(u(x,t)\big) + N\big(u(x,t)\big) + f(x,t) = 0 \qquad (3)$$

With the initial (boundary) conditions

$$B\left(u, \frac{\partial u}{\partial t}\right) = 0 \qquad (4)$$

Where $u(x,t)$ is unknown function, N is nonlinear operator, L is linear operator and f is known function.

The first step of TAM method is assuming that $u_0(x,t)$ is a solution of the eq. (3) of the initial condition

$L\big(u_0(x,t)\big) + f(x,t) = 0$, with $B\left(u_0, \frac{\partial u_0}{\partial t}\right) = 0$

The second iteration can be found by solving the following equation

$L\big(u_1(x,t)\big) + N\big(u_0(x,t)\big) + f(x,t) = 0$, with $B\left(u_1, \frac{\partial u_1}{\partial t}\right) = 0$

After several iterations we can find the general solution of this method which is

$L\big(u_{n+1}(x,t)\big) + N\big(u_n(x,t)\big) + f(x,t) = 0$, with $B\left(u_{n+1}, \frac{\partial u_{n+1}}{\partial t}\right) = 0$

Where u_n is the solution to equation (3) .

Thus, the solution for the equation (3) with (4) is given by

$$u(x,t) = \lim_{n\to\infty} u_n(x,t) \qquad (5)$$

3. The iterative methods of Sine- Gordon equation

In this section, we will present a solution to the Sine-Gordon equation using a modified version of the TAM (Taylor's series and Chebyshev polynomial approximation) method.

3.1 Modified TAM method using Taylor's series

Here, we will apply the third-order Taylor series approximation (TTAM) for $\sin(u)$ in the Sine-Gordon equation and incorporate it into the TTAM method.

First, we express a correction functional for the Sine-Gordon equation (1) by approximating $\sin(u)$ as follows:

$$\sin(u) \approx u - \frac{1}{6}u^3 + \frac{1}{120}u^5. \qquad (6)$$

Thus, equation (1) can be rewritten as:

$$u_{tt}(x,t) - c^2 u_{xx}(x,t) + \alpha u - \frac{1}{6}\alpha u^3 + \frac{1}{120}\alpha u^5 = 0.$$

$$u(x,0) = f(x).$$

$$u_t(x,0) = g(x).$$

. By comparing equation (1) with equation (3), we can express the linear and nonlinear operators as follows:

$$L\big(u(x,t)\big) = u_{tt}(x,t),$$

$$N\big(u(x,t)\big) = -c^2 u_{xx}(x,t) + \alpha u - \frac{1}{6}\alpha u^3 + \frac{1}{120}\alpha u^5, f(x,t) = 0$$

Thus, the initial problem that needs to be solved is:

$$u_{0tt}(x,t) = 0$$

$$u_0(x,0) = f(x)$$

$u_{0t}(x,0) = g(x)$

By integrating the last equation twice from 0 to t with the given initial conditions, we obtain:

$$\int_0^t \int_0^t u_{0tt}(x,t)dtdt = 0.$$

This leads to the expression:

$$u_0(x,t) = f(x) + g(x)t$$

The general solution for the iterative function is given by:

$$\int_0^t \int_0^t u_{(n+1)tt}(x,t)dtdt = \int_0^t \int_0^t \left(c^2 u_{nxx}(x,t) + \alpha u_n - \frac{1}{6}\alpha {u_n}^3 + \frac{1}{120}\alpha {u_n}^5\right) dtdt$$

Therefore, we have:

$$u_1(x,t) = f(x) - \tfrac{1}{2}\alpha t^2 f(x) + \tfrac{1}{12}\alpha t^2 \big(f(x)\big)^3 - \tfrac{1}{240}\alpha t^2 \big(f(x)\big)^5 + tg(x) - \tfrac{1}{6}\alpha t^3 g(x) + \tfrac{1}{12}\alpha t^3 \big(f(x)\big)^2 g(x) - \tfrac{1}{144}\alpha t^3 \big(f(x)\big)^4 g(x) + \tfrac{1}{24}\alpha t^4 f(x)\big(g(x)\big)^2 - \tfrac{1}{144}\alpha t^4 \big(f(x)\big)^3 \big(g(x)\big)^2 + \tfrac{1}{120}\alpha t^5 \big(g(x)\big)^3 - \tfrac{1}{240}\alpha t^5 (f(x))^2 (g(x))^3 - \tfrac{1}{720}\alpha t^6 f(x)(g(x))^4 - \tfrac{1}{5040}\alpha t^7 (g(x))^5 + \tfrac{1}{2}c^2 t^2 f''(x) + \tfrac{1}{6}c^2 t^3 g''(x).$$

$$\vdots$$

We continue the same procedure for the remaining iterations.

3.2 Modified TAM method using Chebyshev polynomial approximation

In the CTAM (Chebyshev's Polynomial) for the Sine-Gordon equation, we will utilize the third Chebyshev's polynomial approximation for$\sin(u)$. By replacing $\sin(u)$in the CTAM with the third Chebyshev's polynomial, we obtain:

$$\sin(u) \approx a + bu + cu^2 + du^3$$

Here, the coefficients are defined as follows:

$$a = \frac{1}{36028797018963968}$$

$$b = \frac{35992149818206681}{36028797018963968}$$

$c = \frac{1}{9007199254740992}$

$d = \frac{1427685159570131}{9007199254740992}$

Hence, equation (1) can be expressed as:

$$u_{tt}(x,t) - c^2 u_{xx}(x,t) + a + \alpha bu + \alpha cu^2 + \alpha du^3 = 0.$$

$$u(x,0) = f(x).$$

$$u_t(x,o) = g(x).$$

Proceeding with the same methodology employed in previous cases, we can derive the subsequent iterations.

4. The convergence analysis of the TAM method:

In this section, we present the condition for convergence of the TAM method [14,15].

Theorem 4.1:

Assume that the series solution $\sum_{i=0}^{\infty} f_i(x,t)$ converges to the solution $u(x,t)$. If the finite series $\sum_{i=0}^{n} f_i(x,t)$ is used as an approximation to the solution of the current equation, then the maximum error $E_n(x,t)$ can be estimated by

$$E_n(x,t) \leq \frac{1}{1-m} m^{n+1} \|f_0\|$$

Corollary 4.2:

The solutions obtained by the Tam method converge to the exact solution under the condition $\exists\ 0 < m < 1\ such\ that\ F[f_0 + f_1 + \cdots + f_{i+1}] \leq mF[f_0 + f_1 + \cdots + f_i]\ (\ i.e\ \|f_{i+1}\| \leq m\|f_i\|)\forall i = 0,1,2,\cdots$. i.e. $\forall i$ if we define the parameters

$$\beta_i = \begin{cases} \frac{\|f_{i+1}\|}{\|f_i\|} & , \|f_i\| \neq 0 \\ 0 & \|f_i\| = 0 \end{cases}$$

Thus, the series solution $\sum_{i=0}^{\infty} f_i(x,t)$ converges to the solution $u(x,t)$, when $0 \le \beta_i < 1 \; \forall i = 0,1,2,\cdots$. Also, as in theorem 4.1 the maximum truncation error is estimated to be
$\| u(x,t) - \sum_{i=0}^{n} f_i(x,t) \| \le \frac{1}{1-\beta} \beta^{n+1} \|f_0\| \; where \; \beta = max\{\beta_i, i = 0,1,2,\cdots,n\}$.

5. Numerical examples

In this section, we explore various examples to demonstrate the effectiveness of the TTAM and CTAM methods in solving the given problems. We apply the TTAM and CTAM techniques to solve these examples, and all computations and numerical analyses are conducted using the Mathematica software.

By employing the TTAM and CTAM methods, we aim to showcase the capabilities of these approaches in providing accurate and efficient solutions to the presented examples. The utilization of Mathematica software ensures reliable and precise calculations, enabling us to obtain reliable numerical results.

Example 5.1

Given the nonlinear Sine-Gordon equation as follows

$$u_{tt} - u_{xx} + \sin(u) = 0 \qquad (7)$$

With the initial conditions

$$u(x,0) = 0, u_t(x,0) = 4\,\mathrm{sech}(x) \qquad (8)$$

And the exact solution given by

$$u(x,t) = 4\tan^{-1}[t\,\mathrm{sech}(x)]$$

By solving eq. (7) with the initial conditions (8) using the same steps used in the TTAM method, we get

$u_0(x,t) = 4t\,\mathrm{sech}(x)$

$u_1(x,t) = (\frac{2}{3}t^3\tanh^2(x) - \frac{2t^3}{3} + 4t)\mathrm{sech}\,(x) + (\frac{8t^5}{15} - \frac{2t^3}{3})\mathrm{sech}^3(x) - \frac{64}{315}t^7\mathrm{sech}^5(x)$

$u_2(x,t) = 4t\text{sech}\,(x) + (\frac{32t^7}{315} - \frac{4t^3}{3})\text{sech}^3(x) + (-\frac{8t^9}{945} - \frac{64t^7}{105} + \frac{4t^5}{5})\text{sech}^5(x) + (\frac{32t^{13}}{8775} - \frac{64t^{11}}{693} + \frac{188t^9}{567})\text{sech}^7(x) + (\frac{16t^{17}}{172125} - \frac{5056t^{15}}{496125} + \frac{2672t^{13}}{36855} - \frac{80t^{11}}{891})\text{sech}^9(x) + (-\frac{8192t^{19}}{12119625} + \frac{17984t^{17}}{1686825} - \frac{9728t^{15}}{297675} + \frac{64t^{13}}{3159})\text{sech}^{11}(x) + \cdots$

⋮

Hence, we can see this series converges to the exact solution when $n \to \infty$

i.e. $u(x,t) = \lim_{n\to\infty} u_n\,(x,t) = 4\tan^{-1}[t\,\text{sech}(x)]$.

We also solve eq. (3.1) with the initial conditions (3.2) by using CTAM method, we obtain

$u_0(x,t) = 4\text{tSech}\,(x)$

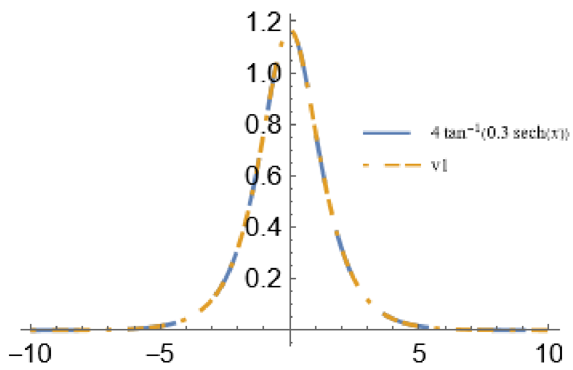
$u_1(x,t) = -\frac{at^2}{2} + \text{sech}\,(x)(-\frac{2bt^3}{3} + \frac{2}{3}t^3\tanh^2(x) + 4t) - \frac{4}{3}ct^4\text{sech}^2(x) + (-\frac{16}{5}dt^5 - \frac{2t^3}{3})\text{sech}^3(x)$

$u_2(x,t) = \frac{1}{448}a^3dt^8 - \frac{1}{120}a^2ct^6 + \frac{1}{24}abt^4 - \frac{at^2}{2} + \Big(-\frac{1}{144}a^2d\tanh^2(x)t^9 + \frac{1}{144}a^2bdt^9 + \frac{1}{63}act^7 - \frac{1}{63}abct^7 - \frac{1}{14}a^2dt^7 + \frac{b^2t^5}{30} + \frac{1}{5}act^5 - \frac{bt^5}{15} + \frac{t^5}{30} + \frac{2}{3}\tanh^2(x)t^3 - \frac{2bt^3}{3} + 4t\Big)\,\text{sech}(x) + (\frac{1}{135}ad\tanh^4(x)t^{10} - \frac{2}{135}abd\tanh^2(x)t^{10} + \frac{1}{135}ab^2dt^{10} + \frac{1}{90}a^2cdt^{10} - \frac{1}{42}ac^2t^8 + \frac{1}{7}ad\tanh^2(x)t^8 - \frac{1}{126}b^2ct^8 + \frac{1}{63}bct^8 - \frac{1}{7}abdt^8 - \frac{ct^8}{126} - \frac{8}{45}c\tanh^2(x)t^6 + \frac{2}{9}bct^6 + \frac{4}{5}adt^6 - \frac{8ct^6}{45} - \frac{4ct^4}{3})\text{sech}^2(x) + \cdots\cdots$

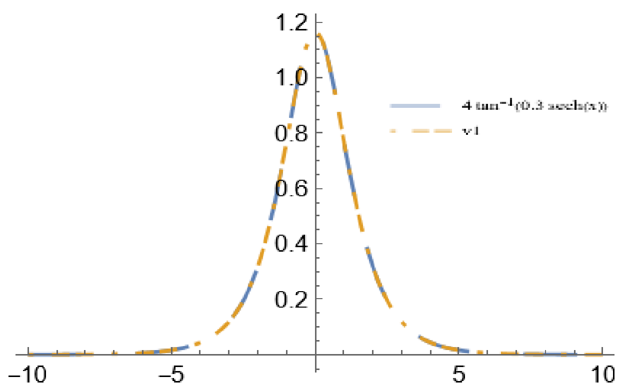
This series converges to the exact solution when

$$u(x,t) = \lim_{n\to\infty} u_n(x,t) = 4\tan^{-1}(t\,\text{sech}(x)).$$

Fig. (1) Shows the comparison between the exact solution and 1-iteration of TTAM and CTAM methods when $-10 \le x \le 10, t = 0.3$



(a)



(b)

Fig. (1). (a) exact solution and TTAM method, (b) exact solution and CTAM method

Example 5.2:

Consider the following nonlinear Sine-Gordon equation

$$u_{tt} - u_{xx} = -\sin(u) \qquad (9)$$

With the initial conditions

$$u(x,0) = \pi + \in \cos(\mu x)\,, u_t(x,0) = 0 \qquad (10)$$

By following the same steps used in Ex.1, we can find :

First, (TTAM)

$$u_0(x,t) = \pi + \in \cos(\mu x)$$

$$u_1(x,t) = \pi + \in \cos(\mu x) + \frac{1}{2}\left(-\epsilon\mu^2 \cos(\mu x) - \frac{1}{120}(\epsilon\cos(\mu x) + \pi)^5 + \frac{1}{6}(\epsilon\cos(\mu x) + \pi)^3 - \epsilon\cos(\mu x) - \pi\right)t^2$$

$$u_2(x,t) = \pi + \in \cos(\mu x) - \frac{\pi t^2}{2} - \frac{1}{2}t^2\epsilon\cos(x\mu) - \frac{1}{2}t^2\epsilon\mu^2\cos(x\mu) - \frac{1}{24}\left(-\frac{1}{120}(\epsilon\cos(x\mu) + \pi)^5 + \frac{1}{6}(\epsilon\cos(x\mu) + \pi)^3 - \epsilon\mu^2\cos(x\mu) - \epsilon\cos(x\mu) - \pi\right)t^4 + \cdots$$

Second (CTAM) we get

$$u_0(x,t) = \pi + \in \cos(\mu x)$$

$$u_1(x,t) = \pi + \epsilon \cos(x\mu) + \frac{1}{2}t^2(-a - b(\epsilon\cos(x\mu) + \pi) - c(\epsilon\cos(x\mu) + \pi)^2 - d(\epsilon\cos(x\mu) + \pi)^3 - \epsilon\mu^2\cos(x\mu))$$

$$u_2(x,t) = \pi + \epsilon \cos(x\mu) - \frac{at^2}{2} - \frac{1}{2}t^2\epsilon\mu^2 \cos(x\mu) - \frac{1}{24}bt^2\big(12\pi + 12\epsilon \cos(x\mu) + t^2(-a - \epsilon\mu^2 \cos(x\mu) - b(\epsilon \cos(x\mu) + \pi) - c(\epsilon \cos(x\mu) + \pi)^2 - d(\epsilon \cos(x\mu) + \pi)^3)\big) + \cdots$$

Now we will use the previous iterations to see that the obtained solution by the TTAM and CTAM methods satisfies the convergence conditions by evaluating the β_i values for examples (1) and (2), we get

In the first method (TTAM) we get

$$\beta_0 = \frac{\|f_1\|}{\|f_0\|} = 0.202976 < 1$$

$$\beta_1 = \frac{\|f_2\|}{\|f_1\|} = 0.131398 < 1$$

$$\beta_2 = \frac{\|f_3\|}{\|f_2\|} = 0.266858 < 1$$

Also, in the second method (CTAM) we have

$$\beta_0 = \frac{\|f_1\|}{\|f_0\|} = 0.381268 < 1$$

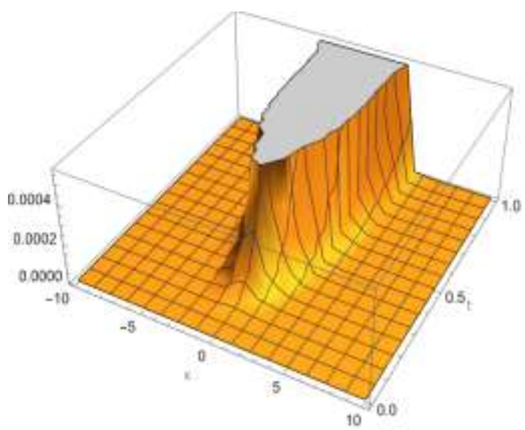
$$\beta_1 = \frac{\|f_2\|}{\|f_1\|} = 0.238958 < 1$$

So, we can see the ex.1 satisfy the convergence, because the β_i values for $i \geq 1$ and $0 < x \leq 1$ are less than 1 when $t = 1$.

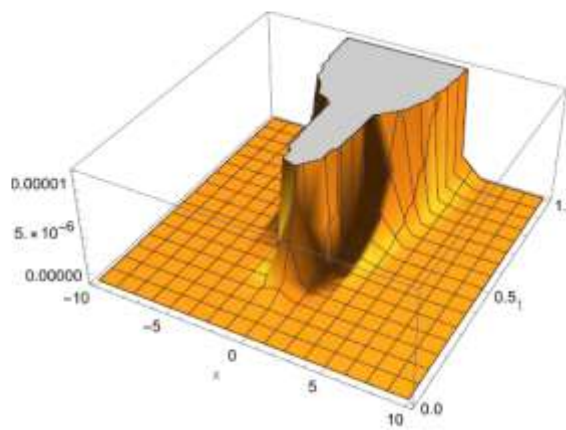
To examine the accuracy the approximate solution, we calculate the absolute error between the exact solution $u(x,t) = 4\tan^{-1}[t\,\mathrm{sech}(x)]$ and approximate solution $u_n(x,t)$. Table (1), figure (2) shows the absolute error of TTAM, CTAM methods and 3D plot of the $|r|$ when $t = 0.3, -10 \leq x \leq 10$ with $n = 1, 2$.

x	$\lvert r_1\rvert$ by $TTAM$	$\lvert r_1\rvert$ by $CTAM$	$\lvert r_2\rvert$ by $TTAM$	$\lvert r_2\rvert$ by $CTAM$
-10	9.70198×10^{-16}	1.66245×10^{-9}	1.6629×10^{-17}	1.66246×10^{-9}
-8	3.91405×10^{-13}	1.22836×10^{-8}	6.70969×10^{-15}	1.22832×10^{-8}
-6	1.57896×10^{-10}	9.0616×10^{-8}	2.70648×10^{-12}	9.04559×10^{-8}
-4	6.35082×10^{-8}	6.09805×10^{-7}	1.08218×10^{-9}	5.45247×10^{-7}
-2	2.17113×10^{-5}	2.08469×10^{-5}	2.52389×10^{-7}	4.28956×10^{-5}
0	5.75612×10^{-4}	3.04414×10^{-3}	1.17881×10^{-5}	2.33028×10^{-3}
2	2.17113×10^{-5}	2.08469×10^{-5}	2.52389×10^{-7}	4.28956×10^{-5}
4	6.35082×10^{-8}	6.09805×10^{-7}	1.08218×10^{-9}	5.45247×10^{-7}
6	1.57896×10^{-10}	9.0616×10^{-8}	2.70648×10^{-12}	9.04559×10^{-8}
8	3.91405×10^{-13}	1.22836×10^{-8}	6.70969×10^{-15}	1.22832×10^{-8}
10	9.70198×10^{-16}	1.66245×10^{-9}	1.6629×10^{-17}	1.66246×10^{-9}

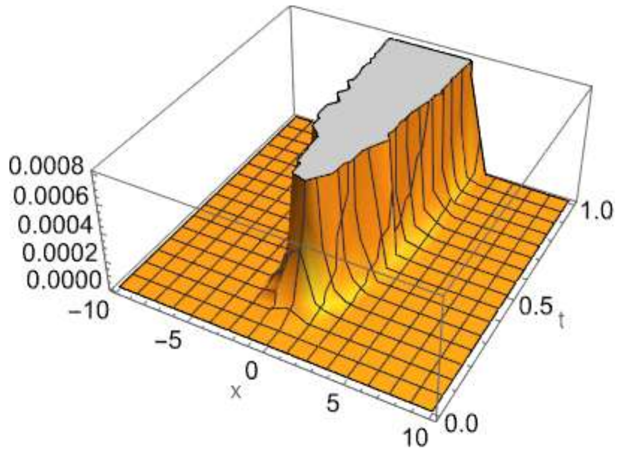
Table (1) results the absolute error by TTAM, CTAM methods when $t = 0.3$



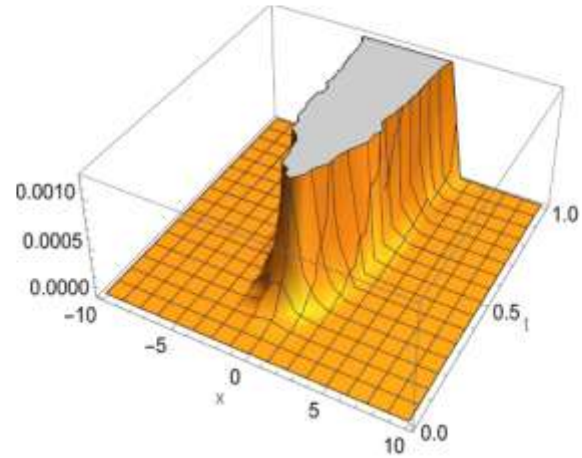
$\lvert r_1\rvert$ by $TTAM$



$\lvert r_2\rvert$ by $TTAM$



$|r_1|$ by $CTAM$



$|r_2|$ by $CTAM$

Fig. (2). the absolute error by TTAM, CTAM methods when $t = 0.3$

In the second example, where $\in = 0.5, \mu = \frac{\sqrt{2}}{2}$, we can see

In TTAM we obtain

$$\beta_0 = \frac{\|f_1\|}{\|f_0\|} = 0.0380656 < 1$$

$$\beta_1 = \frac{\|f_2\|}{\|f_1\|} = 0.03763332 < 1$$

Also, in CTAM we get

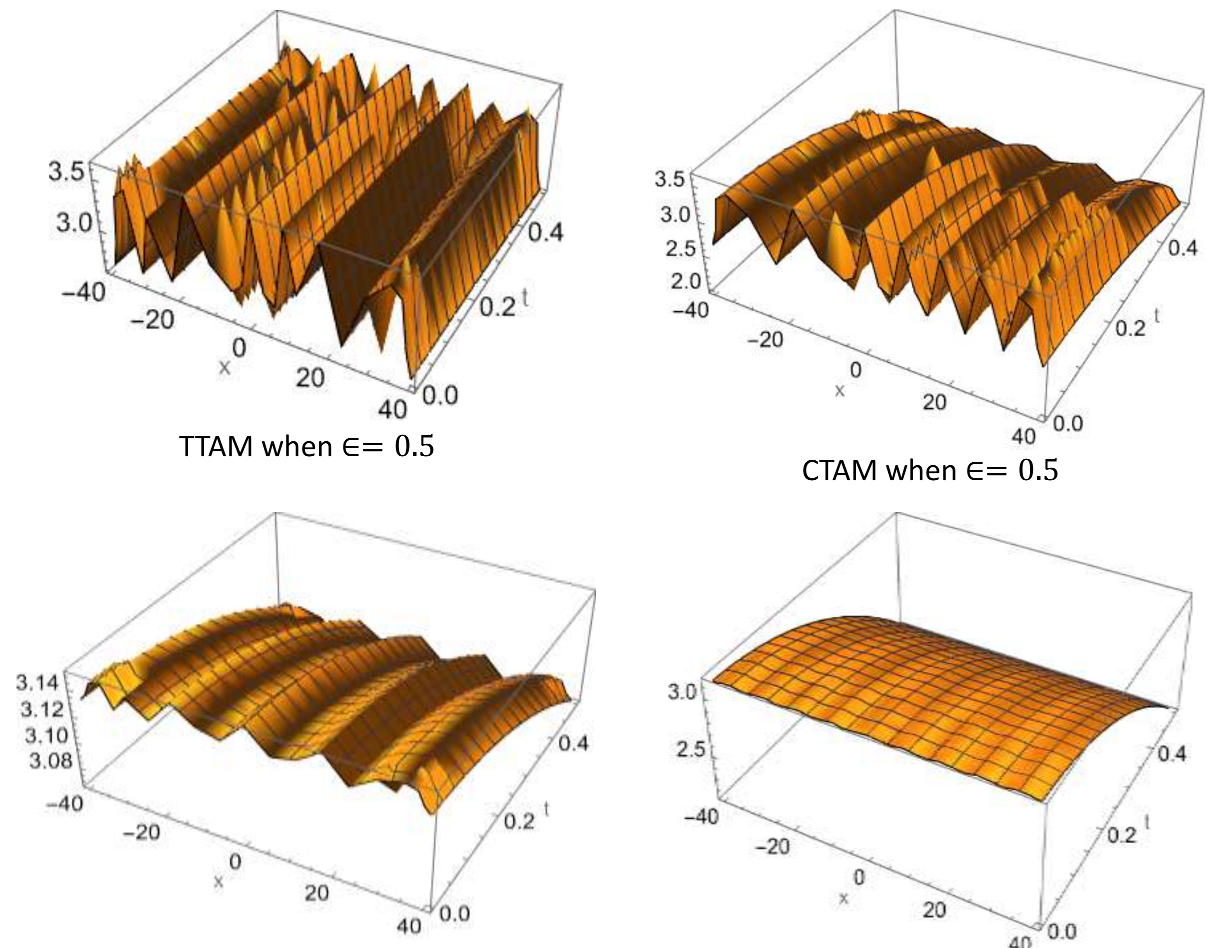
$$\beta_0 = \frac{\|f_1\|}{\|f_0\|} = 0.389665 < 1$$

$$\beta_1 = \frac{\|f_2\|}{\|f_1\|} = 0.133819 < 1$$

Where, the β_i values for $i \geq 1$ and $0 < x \leq 1$ are less than 1 when $t = 0.5$, so the proposed cases are convergent.

Fig. [3]. Shows the comparison between the 1-iterate of two cases when

$\in = 0.5, 0.01$



TTAM when ∈= 0.5

CTAM when ∈= 0.5

TTAM when ∈= 0.01

CTAM when ∈= 0.01

Fig. (3). the 1-iterate of TTAM, CTAM method when ∈= 0.5, 0.01

2. Conclusion

In conclusion, this study presents an investigation into finding an approximate solution to the Sine-Gordon equation through a modification of the Tamimi-Ansari method. By replacing the nonlinear function with a combination of Taylor series and Chebyshev polynomials, this approach offers a significant advantage by eliminating the need for restrictive assumptions when dealing with nonlinear functions.

The obtained numerical results provide strong evidence for the effectiveness of the modified Tamimi-Ansari method. It can be concluded that this method serves as a powerful tool with excellent convergence properties for approximating solutions to the Sine-Gordon equation.

The utilization of Mathematica 13.2 software played a crucial role in performing the calculations and generating the series required by the proposed method. Its robust computational capabilities ensured accuracy and efficiency in the analysis.

The findings of this study contribute to the field of solving nonlinear equations and provide a valuable alternative for obtaining approximate solutions to the Sine-Gordon equation. Further research can explore the application of this method to other nonlinear equations and investigate its limitations and potential extensions.

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