A Simple Algorithm for Summing the Powers of Natural Numbers Using Differential Operator

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ABSTRACT

For a long time, mathematicians have been working on a variety of methods to compute the sum of powers of natural numbers. The relationship between the sum of the kth powers and the sum of the powers $(k-1)$th of the natural numbers will be illustrated in this paper utilizing the differential operator and the associated recurrence relation. Our aim in this research is to find a formula for computing the sum of natural numbers using a combination of geometric series, derivatives, and limits. The findings will be compared with Bernoulli approach. This method distinguishes out because to its uncomplicated computations and clear concepts.

Keywords: differential operator, limits, derivatives, and geometric series

1 Introduction

The power sum of numbers has its applications in number theory and combinatorial mathematics nature. The existing algorithms of computing power sum includes recursive method (see (Ravi PVN, 2021), (Zhu, 2018)). This paper proposes a new algorithm by acting a differential operator $\left(\frac{d}{dx}x\right)^p$ on a geometric series with various powers p , and deriving its formula. The code is supplied below and the results are obtained using Matlab .

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2 Basic Definitions

The fundamental definitions used in this paper are presented in this section. First, geometric series, the derivative, and L'Hôpital's rule are introduced (Andrews, 1998).

Definition 2.1: (GEOMETRIC SERIES)

A **finite geometric series** S_n has one of the following (all equivalent) forms.

$$S_n = a + ax + a\,x^2 + a\,x^3 + a\,x^4 + \dots ++ a\,x^{n-1}$$

$$= \sum_{p=0}^{n-1} a\,x^p$$

Because it is the ratio of consecutive terms in the series, the number x is known as the ratio of the geometric series.

The sum of a finite geometric series is given by:

$$S_n = a + ax + a x^2 + a x^3 + a x^4 + \ldots + + a x^{n-1} = \frac{a(x^n - 1)}{(x - 1)}$$

Definition 2.2: (Derivative of a function)
The derivative of a function $f(x)$ with respect to x is the function $f'(x)$ and is defined as (Habre, 2006):

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Definition 2.3: If at a given point two functions have an infinite limit or zero as a limit and are both differentiable in a neighborhood of this point then the limit of the quotient of the functions is equal to the limit of the quotient of their derivativesthat this limit exists (Fischer, 1983).

3 Calculation to Process Power Sum of Natural Number

The following steps provide at least a sketch of what we mean when evaluating *the powers sum of natural numbers*;

$$S(n, p) = 1^p + 2^p + 3^p + 4^p + 5^p + \ldots + + n^p$$

using *differential operator technique*:

1. Consider the *sum* $S_n(x)$ as a functionof x; where x is known as the ratio of the series $S_n(x)$,

$$S_n(x) = 1 + x + x^2 + x^3 + x^4 + \ldots + + x^{n-1} = \frac{x^n - 1}{x - 1}$$

2. Acting the *differential operator* $\left(\frac{d}{dx} x\right)^p$ on both sides of the equation where $p = 1, 2, 3, \ldots$

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$$\left(\frac{d}{dx} x\right)^p (1 + x + x^2 + x^3 + x^4 + \ldots + + x^{n-1}) = \left(\frac{d}{dx} x\right)^p \frac{x^n - 1}{x - 1}$$

3. Take limit to both sides of the equation as x approaches to 1.

$$\lim_{x \to 1} \left(\frac{d}{dx} x\right)^p (1 + x + x^2 + x^3 + x^4 + \ldots + + x^{n-1}) = \lim_{x \to 1} \left(\frac{d}{dx} x\right)^p \frac{x^n - 1}{x - 1}$$

4. Compute $S(n, p)$ by using the following formula:

$$S(n, p) = 1^p + 2^p + 3^p + 4^p + 5^p + \dots ++ n^p = \lim_{x \to 1} \left(\frac{d}{dx} x\right)^p \frac{x^n - 1}{x - 1}$$

The mathematical induction can be used to prove that:

$$1^p + 2^p + 3^p + 4^p + 5^p + \dots ++ n^p = \lim_{x \to 1} \left(\frac{d}{dx} x\right)^p \frac{x^n - 1}{x - 1} \; ; \; p = 1, 2, 3, \dots$$

Proof:

Let $P(p)$ be the statement that

$$P(p) : 1^p + 2^p + 3^p + 4^p + 5^p + \dots ++ n^p = \lim_{x \to 1} \left(\frac{d}{dx} x\right)^p \frac{x^n - 1}{x - 1}$$

Base Case: When p = 1 the left-hand side of the equation is

$$1^1 + 2^1 + 3^1 + 4^1 + 5^1 + \dots ++ n^1$$

and the right-hand side is

$$\lim_{x \to 1} \left(\frac{d}{dx} x\right)^1 \frac{x^n - 1}{x - 1} = \frac{n(n+1)}{2}$$

So $P(1)$ is correct.

Induction hypothesis: Assume that $P(k)$ is correct for some positive integer k. That means that the left-hand side of the equation equals the right-hand side, so

$$1^k + 2^k + 3^k + 4^k + 5^k + \dots ++ n^k = \lim_{x \to 1} \left(\frac{d}{dx} x\right)^k \frac{x^n - 1}{x - 1}$$

Induction step: We will now show that $P(k + 1)$ is correct by taking the left-hand side of the equation and re-arranging it until it equals the right-hand side. At each stage keep in mind what we are aiming for. So, in this case the right-hand side of the equation should be

since we have replaced the p with (k + 1). So, starting with the left-hand side we have

$$1^{k+1} + 2^{k+1} + 3^{k+1} + 4^{k+1} + 5^{k+1} + \dots ++ n^{k+1} = \lim_{x \to 1} \left(\frac{d}{dx} x\right)^1 \left(\frac{d}{dx} x\right)^k \frac{x^n - 1}{x - 1}$$

$$= \lim_{x \to 1} \left(\frac{d}{dx} x\right)^{1+k} \frac{x^n - 1}{x - 1}$$

$$= \lim_{x \to 1} \left(\frac{d}{dx} x\right)^{k+1} \frac{x^n - 1}{x - 1}$$

So $P\ (k\ +1)$ is correct. Hence by mathematical induction $P\ (p\ )$ is correct for all positive integers p

Therefore,

$$P\ (p\ ):\ 1^p\ +2^p\ +3^p\ +4^p\ +5^p\ +\ldots++n^{\,p} = \lim_{x\to 1}\left(\frac{d}{dx}x\right)^p \frac{x^{\,n}-1}{x\ -1}\ \textit{is}$$

true

The code is supplied below and the results are obtained using MATLAB

```
syms x n k
p=2
f=@(x,n)(x.^(n+1)-1)/(x-1)
if p==1
sp= diff(f(x,n),x)
else
sp= diff(f(x,n),x)
for i=1:p
f= @(x,n)sp
if i==p
sp=f
else
sp=diff(x.*f(x,n),x)
end
end
end
L=@(n)limit(sp,x,1)
```

Table1. Using a simple algorithm to compute power sum of natural numbers

$S(n,p$	$1^p+2^p+3^p+\dots+n^p$	$\lim_{x\to 1}\left(\frac{d}{dx}x\right)^1\frac{x^n-1}{x-1}$	**The Solution as a function of n**
$(n,1)$	$1^1+2^1+3^1+\dots+n^1$	$\lim_{x\to 1}\left(\frac{d}{dx}x\right)^1\frac{x^n-1}{x-1}$	$\frac{\beta}{2}$; $\beta=n(n+1)$
$(n,2)$	$1^2+2^2+3^2+\dots+n^2$	$\lim_{x\to 1}\left(\frac{d}{dx}x\right)^2\frac{x^n-1}{x-1}$	$\frac{\propto}{6}$; $\propto=n(n+1)(2n+1)$
$(n,3)$	$1^3+2^3+3^3+\dots+n^3$	$\lim_{x\to 1}\left(\frac{d}{dx}x\right)^3\frac{x^n-1}{x-1}$	$\frac{\beta^2}{4}$
$(n,4)$	$1^4+2^4+3^4+\dots+n^4$	$\lim_{x\to 1}\left(\frac{d}{dx}x\right)^4\frac{x^n-1}{x-1}$	$\frac{\propto(3n^2+3n-1))}{30}$
$(n,5)$	$1^5+2^5+3^5+\dots+n^5$	$\lim_{x\to 1}\left(\frac{d}{dx}x\right)^5\frac{x^n-1}{x-1}$	$\frac{\beta^2(2n^2+2n-1)}{12}$
$(n,6)$	$1^6+2^6+3^6+\dots+n^6$	$\lim_{x\to 1}\left(\frac{d}{dx}x\right)^6\frac{x^n-1}{x-1}$	$\frac{\propto(3n^4+6n^3-3n+1)}{42}$
$(n,7)$	$1^7+2^7+3^7+\dots+n^7$	$\lim_{x\to 1}\left(\frac{d}{dx}x\right)^7\frac{x^n-1}{x-1}$	$\frac{\beta^2(3n^4+6n^3-n^2-4n+2)}{24}$
$(n,8)$	$1^8+2^8+3^8+\dots+n^8$	$\lim_{x\to 1}\left(\frac{d}{dx}x\right)^8\frac{x^n-1}{x-1}$	$\frac{\propto(5n^6+15n^5+5n^4-15n^3-n^2+9n-3)}{90}$
$(n,9)$	$1^9+2^9+3^9+\dots+n^9$	$\lim_{x\to 1}\left(\frac{d}{dx}x\right)^9\frac{x^n-1}{x-1}$	$\frac{\beta^2(n^2+n-1)(2n^4+4n^3-n^2-3n+3)}{20}$
$(n,10)$	$1^{10}+2^{10}+3^{10}+\dots+n^{10}$	$\lim_{x\to 1}\left(\frac{d}{dx}x\right)^{10}\frac{x^n-1}{x-1}$	$\frac{\propto(n^2+n-1)(3n^6+9n^5+2n^4-11n^3+3n^2+10n-5)}{66}$

In Table 1. We give several examples of power sums of natural numbers that have been solved by applying the method of the differential operator. It was evident that, even with computer software, the differential operator's technique was far easier to apply during the period when it was hard to handle complex concepts in mathematics.

All coefficients can be computed from the recurrence relation given by **(see** (Ravi PVN, 2021)**,** (Zhu, 2018) :

$$\binom{p+1}{i+1}C_0+\binom{p}{i}C_1+\binom{p-1}{i-1}C_2+\dots+\binom{p+i-1}{i}C_i=\binom{p}{i};\qquad i=0,1,2,3,\dots.$$

Where $\binom{m}{n}=\frac{m!}{(m-n)!n!}$;

The table below shows the coefficients of polynomials through the previous formulas where $\left\{\begin{matrix} m \\ n \end{matrix}\right\} = \frac{m!}{(m-n)!}$

$S(n,p)$	$\frac{1}{p+1}$	$\frac{1}{2}$	$\frac{1}{12}\left\{\begin{matrix} p \\ 1 \end{matrix}\right\}$	0	$\frac{-1}{720}\left\{\begin{matrix} p \\ 3 \end{matrix}\right\}$	0	$\frac{1}{30240}\left\{\begin{matrix} p \\ 5 \end{matrix}\right\}$	0	$\frac{-1}{1209600}\left\{\begin{matrix} p \\ 7 \end{matrix}\right\}$	0	$\frac{1}{47900160}\left\{\begin{matrix} p \\ 9 \end{matrix}\right\}$	0
$S(n,1)$	$\frac{1}{2}$	$\frac{1}{2}$										
$S(n,2)$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{6}$									
$S(n,3)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	0								
$S(n,4)$	$\frac{1}{5}$	$\frac{1}{2}$	$\frac{1}{3}$	0	$\frac{-1}{30}$							
$S(n,5)$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{5}{12}$	0	$\frac{-1}{12}$	0						
$S(n,6)$	$\frac{1}{7}$	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{-1}{6}$	0	$\frac{1}{42}$					
$S(n,7)$	$\frac{1}{8}$	$\frac{1}{2}$	$\frac{7}{12}$	0	$\frac{-7}{24}$	0	$\frac{1}{12}$	0				
$S(n,8)$	$\frac{1}{9}$	$\frac{1}{2}$	$\frac{2}{3}$	0	$\frac{-7}{15}$	0	$\frac{2}{9}$	0	$\frac{-1}{30}$			
$S(n,9)$	$\frac{1}{10}$	$\frac{1}{2}$	$\frac{3}{4}$	0	$\frac{-7}{10}$	0	$\frac{1}{2}$	0	$\frac{-3}{20}$	0		
$S(n,10)$	$\frac{1}{11}$	$\frac{1}{2}$	$\frac{5}{6}$	0	-1	0	1	0	$\frac{-1}{2}$	0	$\frac{5}{66}$	0

Conclusions and Remarks

This paper discusses how basic concepts from early university studies, such geometric series and differentiation, may be used as an alternative for ideas from more advanced university courses. However, it is useful for non-mathematicians who struggle to understand complex concepts. That’s why dealing with limits and the differentiation is simpler than dealing with Bernoulli, for instance.

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