CERTAIN SUBCLASSES OF BI-UNIVALENT FUNCTIONS RELATED TO NEW GENERALIZED DERIVATIVE OPERATOR

Farida Abdallah Abufares [1] and Aisha Ahmed Amer [2]

[1] Mathematics Department, Faculty of Science - Alasmarya Islamic University

[2] Mathematics Department, Faculty of Science -Al-Khomus, Al-Margib University

[1]faridaabufares995@gmail.com

[2] aisha.amer@elmergib.edu.ly

Abstract

In this paper, we study bounds for the Taylor–Maclaurin coefficients $|a_2|$ and$|a_3|$, in the open unit disk $\mathbb{D}$. To this purpose, we establish two new subclasses of analytic and univalent functions, in addition to bi-univalent functions associated with a generalized derivative operator.

Keywords

Analytic and univalent functions, Bi-univalent functions, Coefficients bounds, Derivative operator, Starlike and convex functions.

الخلاصة

في هذه الورقة البحثية ‘ندرس حدود معاملات تايلور-ما كلورين $|a_2|$ و $|a_3|$,في قرص الوحدة المفتوح $\mathbb{D}$ ‘ ولهذا الغرض ‘ نتقدم فصلين جزئيين جديدين من الدوال التحليلية أحاديه التكافؤ و ثنائية التكافؤ المرتبطة بمؤثر تفاضلي معمم .

كلمات مفتاحية

الدوال التحليلية أحادية التكافؤ - الدوال التحليلية ثنائية التكافؤ‘ معاملات المحدودة ‘مؤثر التفاضلي والدوال المحدبة والدوال النجمية.

1. Introduction

Let $\mathcal{A}$ be the class of functions$f$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \qquad (1)$$

which are analytic in the open unit disk

$$\mathbb{D} = \{z \in \mathbb{C} \ ; \ |z| < 1\},$$

and satisfy the normalization condition,

$$f(0) = 0 \quad , \quad \grave{f}(0) = 1.$$

The convolution $(*)$ of f and g defined by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k,$$

where,

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k \quad , \qquad (z \in \mathbb{D}).$$

A function f is said to be univalent in $\mathbb{D}$if it is one-to-one in $\mathbb{D}$. we denote by S the subclass of $\mathcal{A}$ consisting of functions that are univalent in $\mathbb{D}$. For example, the Koebe function is in S,

$$\mathcal{K}(z) = \frac{z}{(1-z)^2} = \sum_{k=1}^{\infty} k z^k \ , \quad (z \in \mathbb{D}).\text{(Sabir, 2024)}$$

It is well known that the image of $\mathbb{D}$ under every function $f \in S$ contains a disk of radius $\frac{1}{4}$. Therefore, every function $f \in S$ has an inverse f^{-1}such that

$$f^{-1}\big(f(z)\big) = z \ \ (z \in \mathbb{D})$$

and

$$f\big(f^{-1}(w)\big) = w \ \left(|w| < r_0(f), r_0(f) \geq \frac{1}{4}\right).$$

The inverse function $g = f^{-1}$ has the form

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - \cdots \qquad (2)$$

A function $f \in \mathcal{A}$is said to be bi-univalent function in $\mathbb{D}$. if f andf^{-1} both are univalent in $\mathbb{D}$.(Juma and Aziz, 2012)

The class Σ of all bi-univalent functions in $\mathbb{D}$ is denoted by (1).(Brannan and Taha, 1988) introduced some bi-univalent function class Σ subclasses that are similar to the famous subclasses $S^*(\beta)$.and $C(\beta)$ of starlike and convex functions of order β ($0 < \beta \leq 1$), respectively. (Juma and Aziz, 2012) Thus, a function $f(z) \in \mathcal{A}$ is in the class $S_\Sigma^*(\beta)$ of bi-starlike functions of order β if both f and f^{-1} are starlike functions of order β $(0 < \beta \leq 1)$or into the class $C_\Sigma(\beta)$of bi-convex functions of order β if both f and f^{-1} are convex functions of order β $(o < \beta \leq 1)$. Moreover, A function $f(z) \in \mathcal{A}$ is bi-starlike functions of order β, denoted by the class S_Σ^*, if each of the following conditions is satisfied:

$\left|arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\beta\pi}{2}$ $(0 < \beta \leq 1, z \in \mathbb{D})$ and $\left|arg\left(\frac{wg'(w)}{g(w)}\right)\right| < \frac{\beta\pi}{2}$ $(0 < \beta \leq 1, w \in \mathbb{D})$,

where g is the extension of 1 to $\mathbb{D}$.(Sabir, 2024)

For a function $f(z) \in \mathcal{A}$ defined by (1), the operator $I^m(\lambda_1, \lambda_2, l, n)f(z)$ is defined by $I^m(\lambda_1, \lambda_2, l, n)f(z): \mathcal{A} \to \mathcal{A}$ as follows:

$$I^m(\lambda_1, \lambda_2, l, n)f(z) = z + \sum_{k=2}^{\infty} \frac{(1 + \lambda_1(k-1) + l)^{m-1}}{(1+l)^{m-1}\big(1 + \lambda_2(k-1)\big)^m} c(n,k) a_k z^k,$$

where $n, m \in \mathbb{N}_0 = \{0,1,2, \dots .\}$, $\lambda_2 \geq \lambda_1 \geq 0$, $l \geq 0$, $z \in \mathbb{D}$

$$c(n,k) = \frac{(n+1)_{k-1}}{(1)_{k-1}}. \text{ (Amer and Darus, 2011).}$$

The coefficient estimate problem of Taylor–Maclaurin coefficients,i.e., bound of $|a_k|(k \in \mathbb{N} - \{2, 3\})$ is still an open problem for each $f(z) \in \Sigma$.(Motamednezhad et al., 2022)

The purpose of this paper is to find the coefficient estimate estimates $|a_2|$ and $|a_3|$ for functions in two new subclasses of the function $f(z) \in \Sigma$.

To conclude the results, we need to the following lemma in proving the theorems.

Lemma 1.1:

If $p \in \mathcal{P}$,then $|p_k| \leq 2$ for each $k \in \mathbb{N}$,where $\mathcal{P}$ is the subclass of functions $\mathcal{P}(z)$ of the form

$$\mathcal{P}(z) = 1 + b_1 z + b_2 z^2 + b_3 z^3 + \cdots \qquad (3)$$

Which is analytic in $\mathbb{D}$ and the real part, $Re\ p(z) > 0$ in $\mathbb{D}$ and

$p(0) = 1$.(Duren, 2001)

2. Coefficient Bounds For the Functions in the class $\mathcal{L}_\Sigma(\eta, \omega, m, \alpha)$

A function $p \in \mathcal{P}$ given by (3) and $\mathcal{C}(z)$ be any complex-valued function such that $\mathcal{C}(z) = [\mathcal{P}(z)]^\alpha, 0 < \alpha \leq 1$.

Then

$$|\arg(\mathcal{C}(z))| = \alpha|\arg(h(z)| < \frac{\alpha\pi}{2}.$$

Therefore, if $|\arg(\mathcal{C}(z))| < \frac{\alpha\pi}{2}$, then it can said that there exists $p \in \mathcal{P}$ such that $\mathcal{C}(z)$ can be written in terms of p and α as follows

$\mathcal{C}(z) = [\mathcal{P}(z)]^\alpha$.(Sabir, 2024)

Definition2. 1:

Let $(\eta \geq 0\ , \omega \in \mathbb{C}\backslash\{0\}\ , m \in \mathbb{N}_0\ \ and\ 0 < \ \alpha \leq 1)$.We say that a function $f(z)$ given by (1) is in the class $\mathcal{L}_\Sigma(\eta, \omega, m; \alpha)$ if the following conditions are satisfied:

$$\left|\arg\left(1 + \frac{1}{\omega}\left[(1-\eta)\frac{z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'}{I^m(\lambda_1,\lambda_2,l,n)f(z)} + \eta\frac{\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)' + z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)''}{\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'}\right]\right)\right| < \frac{\alpha\pi}{2} \quad ,$$

$z \in \mathbb{D}$ (4)

and

$$\left|\arg\left(1 + \frac{1}{\omega}\left[(1-\eta)\frac{w\left(I^m(\lambda_1,\lambda_2,l,n)g(w)\right)'}{I^m(\lambda_1,\lambda_2,l,n)g(w)} + \eta\frac{\left(I^m(\lambda_1,\lambda_2,l,n)g(w)\right)' + w\left(I^m(\lambda_1,\lambda_2,l,n)g(w)\right)''}{\left(I^m(\lambda_1,\lambda_2,l,n)g(w)\right)'} - 1\right]\right)\right| < \frac{\alpha\pi}{2}\ , \quad w \in \mathbb{D} \qquad (5)$$

where the function $g = f^{-1}$ is defined by (2).

Marks:

$A = \frac{(1+\lambda_1+l)^{m-1}}{(1+l)^{m-1}(1+\lambda_2)^m}(n+1) \quad , B = \frac{(1+\lambda_1(2)+l)^{m-1}}{(1+l)^{m-1}\left(1+\lambda_2(2)\right)^m}\frac{(n+2)(n+1)}{2}$

Theorem2. 1:

Let $f \in \mathcal{L}_\Sigma(\eta, \omega, m, \alpha)$ be given by (1). Then

$$|a_2| \leq \frac{2\alpha|\omega|}{\sqrt{|2\alpha\omega(2(1+2\eta)\mathrm{B} - (1+3\eta)\mathrm{A}^2) - (\alpha-1)(1+\eta)^2\mathrm{A}^2|}} \tag{6}$$

and

$$|a_3| \leq \frac{4\alpha^2|\omega|^2}{(1+\eta)^2\mathrm{A}^2} + \frac{\alpha|\omega|}{(1+2\eta)\mathrm{B}} \tag{7}$$

Proof

From the definition we get

$$1 + \frac{1}{\omega}\left[(1-\eta)\frac{z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'}{I^m(\lambda_1,\lambda_2,l,n)f(z)} + \eta\frac{\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)' + z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)''}{\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'}\right] = [h(z)]^\alpha \tag{8}$$

and

$$1 + \frac{1}{\omega}\left[(1-\eta)\frac{w\left(I^m(\lambda_1,\lambda_2,l,n)g(w)\right)'}{I^m(\lambda_1,\lambda_2,l,n)g(w)} + \eta\frac{\left(I^m(\lambda_1,\lambda_2,l,n)g(w)\right)' + w\left(I^m(\lambda_1,\lambda_2,l,n)g(w)\right)''}{\left(I^m(\lambda_1,\lambda_2,l,n)g(w)\right)'} - 1\right] = [q(w)]^\alpha \tag{9}$$

where $h, q \in \mathcal{P}$ have the formula as follows

$$h(z) = 1 + h_1 z + h_2 z^2 + h_3 z^3 + \cdots \tag{10}$$

and

$$q(w) = 1 + q_1 w + q_2 w^2 + q_3 w^3 + \cdots \tag{11}$$

then

$$[h(z)]^\alpha = \sum_{k=0}^{\infty}\binom{\alpha}{k}1^{\alpha-k}\left(\sum_{n=1}^{\infty}h_n z^n\right)^k$$

$$[h(z)]^\alpha = \binom{\alpha}{0}1^\alpha + \binom{\alpha}{1}1^{\alpha-1}\sum_{n=1}^{\infty}h_n z^n + \binom{\alpha}{2}1^{\alpha-2}\left(\sum_{n=1}^{\infty}h_n z^n\right)^2 + \binom{\alpha}{3}1^{\alpha-3}\left(\sum_{n=1}^{\infty}h_n z^n\right)^3 + \cdots$$

$$[h(z)]^\alpha = \binom{\alpha}{0}1^\alpha + \binom{\alpha}{1}1^{\alpha-1}(h_1 z + h_2 z^2 + h_3 z^3 + \cdots) + \binom{\alpha}{2}1^{\alpha-2}(h_1^2 z^2 + h_1 h_2 z^3 + \cdots + h_1 h_2 z^3 + \cdots) + \cdots$$

$$[h(z)]^\alpha = 1 + \alpha h_1 z + \alpha h_2 z^2 + \alpha h_3 z^3 + \cdots + \frac{\alpha(\alpha-1)}{2!}h_1^2 z^2 + \frac{\alpha(\alpha-1)}{2!}h_1 h_2 z^3 + \cdots + \frac{\alpha(\alpha-1)}{2!}h_1 h_2 z^3 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}h_1^3 z^3 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}h_1^2 h_2 z^4 + \cdots$$

$[h(z)]^{\alpha} = 1 + \alpha h_1 z + \left[\alpha h_2 + \frac{\alpha(\alpha-1)}{2!} h_1^2\right] z^2 + \left[\alpha h_3 + \frac{2\alpha(\alpha-1)}{2!} h_1 h_2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!} h_1^3\right] z^3 + \cdots$

$[h(z)]^{\alpha} = 1 + \alpha h_1 z + \left[\alpha h_2 + \frac{\alpha(\alpha-1)}{2!} h_1^2\right] z^2 + \left[\alpha h_3 + \alpha(\alpha-1) h_1 h_2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!} h_1^3\right] z^3 + \cdots \quad (12)$

and

$$[q(w)]^{\alpha} = \sum_{k=0}^{\infty} \binom{\alpha}{k} 1^{\alpha-k} \left(\sum_{n=1}^{\infty} q_n w^n\right)^k$$

$[q(w)]^{\alpha} = \binom{\alpha}{0} 1^{\alpha} + \binom{\alpha}{1} 1^{\alpha-1} \sum_{n=1}^{\infty} q_n w^n + \binom{\alpha}{2} 1^{\alpha-2} (\sum_{n=1}^{\infty} q_n w^n)^2 + \binom{\alpha}{3} 1^{\alpha-3} (\sum_{n=1}^{\infty} q_n w^n)^3 + \cdots$

$[q(w)]^{\alpha} = 1 + \alpha q_1 w + \left[\alpha q_2 + \frac{\alpha(\alpha-1)}{2!} q_1^2\right] w^2 + \left[\alpha q_3 + \frac{2\alpha(\alpha-1)}{2!} q_1 q_2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!} q_1^3\right] w^3 + \cdots$

$[q(w)]^{\alpha} = 1 + \alpha q_1 w + \left[\alpha q_2 + \frac{\alpha(\alpha-1)}{2!} q_1^2\right] w^2 + \left[\alpha q_3 + \alpha(\alpha-1) q_1 q_2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!} q_1^3\right] w^3 + \cdots \quad (13)$

when substituting in the class we get

$1 + \frac{1}{\omega}\left[(1-\eta)\frac{z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'}{I^m(\lambda_1,\lambda_2,l,n)f(z)} + \eta\left(1 + \frac{z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)''}{\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'}\right) - 1\right] = 1 + \frac{1}{\omega}\ [1 - \eta + (1-\eta)\mathrm{A}\, a_2 z + (1-\eta)(2\mathrm{B}a_3 - \mathrm{A}^2 a_2^2)\ z^2 + \cdots + \eta + 2\eta\mathrm{A}\, a_2 z + \eta(6\mathrm{B}a_3 - 4\mathrm{A}^2 a_2^2)\ z^2\ + \cdots - 1]\ ,$

$1 + \frac{1}{\omega}\left[(1-\eta)\frac{z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'}{I^m(\lambda_1,\lambda_2,l,n)f(z)} + \eta\left(1 + \frac{z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)''}{\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'}\right) - 1\right] = 1 + \frac{(1+\eta)}{\omega}\mathrm{A}\, a_2 z + \left[\frac{2(1+2\eta)\ \mathrm{B}}{\omega} a_3 - \frac{(1+3\eta)}{\omega}\mathrm{A}^2\ a_2^2\right] z^2\ + \cdots \quad (14)$

Now, we defined the inverse

$$I^m(\lambda_1,\lambda_2,l,n)g(w) = w - \mathrm{A}\ a_2 w^2 + \mathrm{B}(2a_2^2 - a_3)w^3 - \cdots$$

When you do some simple operations, we get

$1 + \frac{1}{\omega}\left[(1-\eta)\frac{z\left(I^m(\lambda_1,\lambda_2,l,n)g(w)\right)'}{I^m(\lambda_1,\lambda_2,l,n)g(w)} + \eta\left(\frac{\left(I^m(\lambda_1,\lambda_2,l,n)g(w)\right)' + z\left(I^m(\lambda_1,\lambda_2,l,n)g(w)\right)''}{\left(I^m(\lambda_1,\lambda_2,l,n)g(w)\right)'}\right) - 1\right] = 1 + \frac{1}{\omega}[1 - \eta - (1-\eta)\mathrm{A}a_2 w + (1-\eta)(2\mathrm{B}(2a_2^2 - a_3) - \mathrm{A}^2 a_2^2)w^2 + \cdots + \eta - 2\eta\mathrm{A}\ a_2 w + \eta(6\mathrm{B}(2a_2^2 - a_3) - 4\mathrm{A}^2 a_2^2)w^2 + \cdots - 1],$

$$1+\frac{1}{\omega}\left[(1-\eta)\frac{z\left(I^m(\lambda_1,\lambda_2,l,n)g(w)\right)'}{I^m(\lambda_1,\lambda_2,l,n)g(w)}+\eta\left(\frac{\left(I^m(\lambda_1,\lambda_2,l,n)g(w)\right)'+z\left(I^m(\lambda_1,\lambda_2,l,n)g(w)\right)''}{\left(I^m(\lambda_1,\lambda_2,l,n)g(w)\right)'}\right)-1\right]=1+\frac{1}{\omega}[-(1-\eta)\mathrm{A}a_2w+(4(1+2\eta)\mathrm{B}a_2^2-2(1+2\eta)\mathrm{B}a_3-(1+3\eta)\mathrm{A}^2\,a_2^2)w^2+\cdots],$$

$$1+\frac{1}{\omega}\left[(1-\eta)\frac{z\left(I^m(\lambda_1,\lambda_2,l,n)g(w)\right)'}{I^m(\lambda_1,\lambda_2,l,n)g(w)}+\eta\left(\frac{\left(I^m(\lambda_1,\lambda_2,l,n)g(w)\right)'+z\left(I^m(\lambda_1,\lambda_2,l,n)g(w)\right)''}{\left(I^m(\lambda_1,\lambda_2,l,n)g(w)\right)'}\right)-1\right]=1-\frac{(1-\eta)}{\omega}\mathrm{A}a_2w+\left[\frac{4(1+2\eta)\mathrm{B}}{\omega}a_2^2-\frac{2(1+2\eta)\mathrm{B}}{\omega}a_3-\frac{(1+3\eta)\mathrm{A}^2}{\omega}\,a_2^2\right]w^2+\cdots \quad (15)$$

Coefficients equal to (12), (14)

$$\frac{(1+\eta)}{\omega}\mathrm{A}a_2=\alpha h_1 \quad (16)$$

$$\frac{2(1+2\eta)\mathrm{B}}{\omega}a_3-\frac{(1+3\eta)}{\omega}\,\mathrm{A}^2a_2^2=\alpha h_2+\frac{\alpha(\alpha-1)}{2!}h_1^2 \quad (17)$$

from (13),(15) we get

$$-\frac{(1+\eta)}{\omega}\mathrm{A}a_2=\alpha q_1 \quad (18)$$

$$\frac{4(1+2\eta)\mathrm{B}}{\omega}a_2^2-\frac{2(1+2\eta)\mathrm{B}}{\omega}a_3-\frac{(1+3\eta)\mathrm{A}^2}{\omega}a_2^2=\alpha q_2+\frac{\alpha(\alpha-1)}{2!}q_1^2 \quad (19)$$

Clearly, from (16),(18) we have

$$h_1=-q_1, \quad (20)$$

we square and addition (16),(18) together ,we get

$$\frac{2(1+\eta)^2\mathrm{A}^2}{\omega^2}a_2^2=\alpha^2(h_1^2+q_1^2), \quad (21)$$

we addition (17),(19) we get

$$\frac{4(1+2\eta)\mathrm{B}}{\omega}a_2^2-\frac{2(1+3\eta)\mathrm{A}^2}{\omega}a_2^2=\frac{1}{2}\alpha(\alpha-1)(h_1^2+q_1^2)+\alpha(h_2+q_2), \quad (22)$$

from (21) substituting$(h_1^2+q_1^2)$ in (22)

$$\frac{4(1+2\eta)\mathrm{B}}{\omega}a_2^2-\frac{2(1+3\eta)\mathrm{A}^2}{\omega}a_2^2-\frac{1}{2}\alpha(\alpha-1)\left[\frac{2(1+\eta)^2\mathrm{A}^2}{\alpha^2\omega^2}a_2^2\right]=\alpha(h_2+q_2)\,.$$

Now

$$\frac{4\alpha\omega(1+2\eta)\mathrm{B}}{\alpha\omega^2}a_2^2-\frac{2\alpha\omega(1+3\eta)\mathrm{A}^2}{\alpha\omega^2}a_2^2-\frac{(\alpha-1)(1+\eta)^2\mathrm{A}^2}{\alpha\omega^2}a_2^2=\alpha(h_2+q_2),$$

$$a_2^2\left[\frac{\alpha\omega(4(1+2\eta)\mathrm{B}-2(1+3\eta)\mathrm{A}^2)-(\alpha-1)(1+\eta)^2\mathrm{A}^2}{\alpha\omega^2}\right]=\alpha(h_2+q_2),$$

$$a_2^2=\frac{\alpha^2\omega^2(h_2+q_2)}{2\alpha\omega\left(2(1+2\eta)\mathrm{B}-(1+3\eta)\mathrm{A}^2\right)-(\alpha-1)(1+\eta)^2\mathrm{A}^2}, \quad (23)$$

from lemma 1.1 for coefficientsh_2 and $q_2$on (23) imply that

$$|a_2| \leq \frac{2\alpha|\omega|}{\sqrt{\left|2\alpha\omega\left(2(1+2\eta)\mathrm{B}-(1+3\eta)\mathrm{A}^2\right)-(\alpha-1)(1+\eta)^2\mathrm{A}^2\right|}},$$

Next, we exist $|a_3|$, by subtracting (19) from (17), we get

$$\frac{4(1+2\eta)\mathrm{B}}{\omega}a_3-\frac{4(1+2\eta)\mathrm{B}}{\omega}a_2^2=\frac{1}{2}\alpha(\alpha-1)(h_1^2-q_1^2)+\alpha(h_2-q_2)\ , \qquad (24)$$

from (21) we substituting a_2^2 in (24) and we put $h_1^2=q_1^2$ from (20)

$$\frac{4(1+2\eta)\mathrm{B}}{\omega}a_3-\frac{4(1+2\eta)\mathrm{B}}{\omega}\left[\frac{\alpha^2\omega^2(h_1^2+q_1^2)}{2(1+\eta)^2\mathrm{A}^2}\right]=\frac{1}{2}\alpha(\alpha-1)(h_1^2-h_1^2)+\alpha(h_2-q_2),$$

$$a_3=\frac{\alpha^2\omega^2(h_1^2+q_1^2)}{2(1+\eta)^2\mathrm{A}^2}+\frac{\alpha\omega(h_2-q_2)}{4(1+2\eta)\mathrm{B}} \qquad (25)$$

Now, applying lemma 1.1 for coefficients h_1, h_2, q_1 and q_2 on (25) we get

$$|a_3| \leq \frac{4\alpha^2|\omega|^2}{(1+\eta)^2\mathrm{A}^2}+\frac{\alpha|\omega|}{(1+2\eta)\mathrm{B}}. \quad \blacksquare$$

Corollary:

If $m=1$, $\lambda_2=0$ .Let f given by (1) be in the class $\mathcal{L}_\Sigma(\eta,\omega,\ell,\alpha)$. Then

$$|a_2| \leq \frac{2\alpha|\omega|}{\sqrt{\left|2\alpha\omega\left((1+2\eta)(\ell+2)(\ell+1)-(1+3\eta)(\ell+1)^2\right)-(\alpha-1)(1+\eta)^2(\ell+1)^2\right|}}$$

and

$$|a_3| \leq \frac{4\alpha^2|\omega|^2}{(1+\eta)^2(\ell+1)^2}+\frac{2\alpha|\omega|}{(1+2\eta)(\ell+2)(\ell+1)}.$$

(Sabir, 2024)

3. Coefficient Bounds For the Functions in the class$\mathcal{L}_\Sigma^*(\eta,\omega,m,\beta)$

A function $p \in \mathcal{P}$ given by (3) and $\mathcal{H}(z)$ be any complex-valued function such that $\mathcal{H}(z)=\beta+(1-\beta)p(z),\ \ 0<\beta\leq 1,$

Then

$$Re\,\{\mathcal{H}(z)\}=\beta+(1-\beta)Re\,\{p(z)\}>\beta.$$

Therefore, if $Re\,\{\mathcal{H}(z)\}>\beta$, then it can be said that there exists $p\in\mathcal{P}$ such that $\mathcal{H}(z)$ can be written in terms of p and β as follows

$$\mathcal{H}(z)=\beta+(1-\beta)p(z).$$

(Sabir, 2024)

Definition3. 1:

Let $(\eta \geq 0\,, \omega \in \mathbb{C}\backslash\{0\}\,, m \in \mathbb{N}_0 \;\; and\; 0 < \beta \leq 1)$.We say that a function $f(z)$ given by (1) is in the class $\mathcal{L}_\Sigma^*(\eta, \omega, m, \beta)$ if the following conditions are satisfied:

$$Re\left\{1 + \frac{1}{\omega}\left[(1-\eta)\frac{z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'}{I^m(\lambda_1,\lambda_2,l,n)f(z)} + \eta\frac{\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)' + z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)''}{\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'} - 1\right]\right\} > \beta \quad , z \in \mathbb{D} \tag{26}$$

and

$$Re\left\{1 + \frac{1}{\omega}\left[(1-\eta)\frac{w\left(I^m(\lambda_1,\lambda_2,l,n)g(w)\right)'}{I^m(\lambda_1,\lambda_2,l,n)g(w)} + \eta\frac{\left(I^m(\lambda_1,\lambda_2,l,n)g(w)\right)' + w\left(I^m(\lambda_1,\lambda_2,l,n)g(w)\right)''}{\left(I^m(\lambda_1,\lambda_2,l,n)g(w)\right)'} - 1\right]\right\} > \beta\,, w \in \mathbb{D} \tag{27}$$

where the function $g = f^{-1}$ is defined by (2).

Theorem3. 1:

Let $f \in \mathcal{L}_\Sigma^*(\eta, \omega, m; \beta)$be given by (1). Then

$$|a_2| \leq \begin{cases} \sqrt{\frac{2|\omega|(1-\beta)}{2(1+2\eta)\mathrm{B}-(1+3\eta)\mathrm{A}^2}} \;, & 0 \leq \beta \leq 1 - \frac{(1+\eta)^2\mathrm{A}^2}{2|\omega|[2(1+2\eta)\mathrm{B}-(1+3\eta)\mathrm{A}^2]} \\ \frac{2|\omega|(1-\beta)}{(1+\eta)\mathrm{A}} \;, & 1 - \frac{(1+\eta)^2\mathrm{A}^2}{2|\omega|[2(1+2\eta)\mathrm{B}-(1+3\eta)\mathrm{A}^2]} \leq \beta < 1 \end{cases}$$

and

$$|a_3| \leq \begin{cases} \frac{2|\omega|(1-\beta)}{2(1+2\eta)\mathrm{B}-(1+3\eta)\mathrm{A}^2} + \frac{|\omega|(1-\beta)}{(1+2\eta)\mathrm{B}} \;, & 0 \leq \beta \leq 1 - \frac{(1+\eta)^2\mathrm{A}^2}{2|\omega|[2(1+2\eta)\mathrm{B}-(1+3\eta)\mathrm{A}^2]} \\ \frac{4|\omega|^2(1-\beta)^2}{(1+\eta)^2\mathrm{A}^2} + \frac{|\omega|(1-\beta)}{(1+2\eta)\mathrm{B}} \;, & 1 - \frac{(1+\eta)^2\mathrm{A}^2}{2|\omega|[2(1+2\eta)\mathrm{B}-(1+3\eta)\mathrm{A}^2]} \leq \beta < 1 \end{cases}$$

Proof

From the definition we get

$$1 + \frac{1}{\omega}\left[(1-\eta)\frac{z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'}{I^m(\lambda_1,\lambda_2,l,n)f(z)} + \eta\frac{\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)' + z\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)''}{\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)'} - 1\right] = \beta + (1-\beta)h(z) \tag{28}$$

and

$$1 + \frac{1}{\omega}\left[(1-\eta)\frac{w\left(I^m(\lambda_1,\lambda_2,l,n)g(w)\right)'}{I^m(\lambda_1,\lambda_2,l,n)g(w)} + \eta\frac{\left(I^m(\lambda_1,\lambda_2,l,n)g(w)\right)' + w\left(I^m(\lambda_1,\lambda_2,l,n)g(w)\right)''}{\left(I^m(\lambda_1,\lambda_2,l,n)g(w)\right)'} - 1\right] = \beta + (1-\beta)q(z), \tag{29}$$

where $h, q \in \mathcal{P}$.Then

$$\beta + (1-\beta)h(z) = \beta + (1-\beta)[1 + h_1 z + h_2 z^2 + h_3 z^3 + \cdots]$$

$$= \beta + 1 - \beta + (1-\beta)h_1 z + (1-\beta)h_2 z^2 + \cdots$$

$$= 1 + (1-\beta)h_1 z + (1-\beta)h_2 z^2 + \cdots \qquad (30)$$

and

$$\beta + (1-\beta)q(z) = \beta + (1-\beta)[1 + q_1 z + q_2 z^2 + q_3 z^3 + \cdots]$$

$$= \beta + 1 - \beta + (1-\beta)q_1 z + (1-\beta)q_2 z^2 + \cdots$$

$$= 1 + (1-\beta)q_1 z + (1-\beta)q_2 z^2 + \cdots \qquad (31)$$

Now, equal coefficients (14) with (30)for h

$$\frac{(1+\eta)\mathrm{A}}{\omega} a_2 = (1-\beta)h_1 \qquad (32)$$

$$\frac{2(1+2\eta)\mathrm{B}}{\omega} a_3 - \frac{(1+3\eta)\mathrm{A}^2}{\omega}\ a_2^2 = (1-\beta)h_2 \qquad (33)$$

equal coefficients (15) with (31)for q

$$-\frac{(1+\eta)\mathrm{A}}{\omega} a_2 = (1-\beta)q_1 \qquad (34)$$

$$\frac{4(1+2\eta)\mathrm{B}}{\omega} a_2^2 - \frac{(1+3\eta)\mathrm{A}^2}{\omega} a_2^2 - \frac{2(1+2\eta)\mathrm{B}}{\omega} a_3 = (1-\beta)q_2 \qquad (35)$$

From (32) and (34) we get

$$h_1 = -q_1 \qquad (36)$$

We square (32) and (34) then addition we get

$$\frac{2(1+\eta)^2\mathrm{A}^2}{\omega^2} a_2^2 = (1-\beta)^2\left({h_1}^2 + {q_1}^2\right) \qquad (37)$$

we add (33) with (35) we get

$$\left[\frac{4(1+2\eta)\mathrm{B}}{\omega} - \frac{2(1+3\eta)\mathrm{A}^2}{\omega}\right] a_2^2 = (1-\beta)(h_1 + q_1) \qquad (38)$$

from (37) we defined

$$a_2^2 = \frac{\omega^2(1-\beta)^2\left({h_1}^2 + {q_1}^2\right)}{2(1+\eta)^2\mathrm{A}^2} \qquad (39)$$

and from (38) we get

$$a_2^2 = \frac{\omega(1-\beta)(h_2 + q_2)}{4(1+2\eta)\mathrm{B} - 2(1+3\eta)\mathrm{A}^2} \qquad (40)$$

the equations (39) and (40) together with applying Lemma 1.1 for the coefficients $h_1, q_1, h_2,$ and q_2, we find that

$$a_2^2 = \frac{4\omega^2(1-\beta)^2}{(1+\eta)^2\mathrm{A}^2} \ \Rightarrow |a_2| \le \frac{2|\omega|(1-\beta)}{(1+\eta)\mathrm{A}},$$

and

$$a_2^2 = \frac{4\omega(1-\beta)}{4(1+2\eta)\mathrm{B} - 2(1+3\eta)\mathrm{A}^2} \Rightarrow a_2^2 = \frac{4\omega(1-\beta)}{2(2(1+2\eta)\mathrm{B} - (1+3\eta)\mathrm{A}^2)}$$

$$a_2^2 = \frac{2\omega(1-\beta)}{2(1+2\eta)\mathrm{B} - (1+3\eta)\mathrm{A}^2}$$

$$|a_2| \le \sqrt{\frac{2|\omega|(1-\beta)}{|2(1+2\eta)\mathrm{B} - (1+3\eta)\mathrm{A}^2|}}$$

respectively.

To determine the estimates on $|a_3|$, by subtracting (35) from (33), we get

$$\frac{4(1+2\eta)\mathrm{B}}{\omega} a_3 - \frac{4(1+2\eta)\mathrm{B}}{\omega} a_2^2 = (1-\beta)(h_2 - q_2),$$

$$a_3 = a_2^2 + \frac{\omega(1-\beta)(h_2-q_2)}{4(1+2\eta)\mathrm{B}} \qquad (41)$$

Substituting the value of a_2^2 from (39) into (41) we get

$$a_3 = \frac{\omega^2(1-\beta)^2\left({h_1}^2 + {q_1}^2\right)}{2(1+\eta)^2\mathrm{A}^2} + \frac{\omega(1-\beta)(h_2-q_2)}{4(1+2\eta)\mathrm{B}} \qquad (42)$$

Substituting the value of a_2^2 from (40) into (41), we get

$$a_3 = \frac{\omega(1-\beta)(h_2+q_2)}{4(1+2\eta)\mathrm{B} - 2(1+3\eta)\mathrm{A}^2} + \frac{\omega(1-\beta)(h_2-q_2)}{4(1+2\eta)\mathrm{B}}, \qquad (43)$$

respectively.

Finally, applying lemma 1.1 for the coefficients $h_2, h_2 + q_1$, and q_2 on equations (42) and (43) together, we conclude that

$$|a_3| \le \frac{4|\omega|^2(1-\beta)^2}{(1+\eta)^2\mathrm{A}^2} + \frac{|\omega|(1-\beta)}{(1+2\eta)\mathrm{B}}$$

and

$$|a_3| \le \frac{2|\omega|(1-\beta)}{|2(1+2\eta)\mathrm{B} - (1+3\eta)\mathrm{A}^2|} + \frac{|\omega|(1-\beta)}{(1+2\eta)\mathrm{B}} \qquad \blacksquare$$

Corollary:

If $m = 1$, $\lambda_2 = 0$ .Let f given by (1) be in the class $\mathcal{L}_\Sigma^*(\eta, \omega, n, \alpha)$. Then

$$|a_2| \le \begin{cases} \sqrt{\dfrac{2|\omega|(1-\beta)}{|(1+2\eta)(n+2)(n+1)-(1+3\eta)(n+1)^2|}}, & 0 \le \beta \le 1 - \dfrac{(1+\eta)^2(n+1)^2}{2|\omega||(1+2\eta)(n+2)(n+1)-(1+3\eta)(n+1)^2|} \\ \dfrac{2|\omega|(1-\beta)}{(1+\eta)(n+1)}, & 1 - \dfrac{(1+\eta)^2(n+1)^2}{2|\omega||(1+2\eta)(n+2)(n+1)-(1+3\eta)(n+1)^2|} \le \beta < 1 \end{cases}$$

and

$$|a_3| \le \begin{cases} \dfrac{2|\omega|(1-\beta)}{|(1+2\eta)(n+2)(n+1)-(1+3\eta)(n+1)^2|} + \dfrac{2|\omega|(1-\beta)}{(1+2\eta)(n+2)(n+1)}, & 0 \le \beta \le 1 - \dfrac{(1+\eta)^2(n+1)^2}{2|\omega||(1+2\eta)(n+2)(n+1)-(1+3\eta)(n+1)^2|} \\ \dfrac{4|\omega|^2(1-\beta)^2}{(1+\eta)^2(n+1)^2} + \dfrac{2|\omega|(1-\beta)}{(1+2\eta)(n+2)(n+1)}, & 1 - \dfrac{(1+\eta)^2(n+1)^2}{2|\omega||(1+2\eta)(n+2)(n+1)-(1+3\eta)(n+1)^2|} \le \beta < 1 \end{cases}$$

(Sabir, 2024)

There are many other works in these references(Abufares and Amer, 2024, Alabbar et al., 2023, Almarwm et al., 2024, Amer, 2016, Amer and Alabbar, 2017, Shmella and Amer, 2024a, Shmella and Amer, 2024b) on analytic functions and univalent function associated by generalized derivative operator.

Conclusion:

In this study, we used two new subclasses of an analytic function and bi-univalent function and defined by generalized derivative operator $I^m(\lambda_1, \lambda_2, l, n)f(z)$ on the open disk, from these new subclasses we obtained on estimates of coefficient bounds for the functions

References

ABUFARES, F. A. & AMER, A. A. 2024. Certain Applications of Analytic Functions Associated in Complex BB Differential Equations. 1 ,مجلة كلية التربية طرابلس.

ALABBAR, N., DARUS, M. & AMER, A. 2023. Coefficient Inequality and Coefficient Bounds for a New Subclass of Bazilevic Functions. *Journal of Humanitarian and Applied Sciences,* 8**,** 496-506.

ALMARWM, H. M., AMER, A. A. & AJAIB, S. K. 2024. The Coefficient Estimates for A New Class of Analytic Functions with Negative Coefficients. -322 ,مجلة جامعة بني وليد للعلوم الإنسانية والتطبيقية 329.

AMER, A. A. Second Hankel Determinant for New Subclass Defined by a Linear Operator. Computational Analysis: AMAT, Ankara, May 2015 Selected Contributions, 2016. Springer, 79-86.

AMER, A. A. & ALABBAR, N. M. 2017. Properties of Generalized Derivative Operator to A Certain Subclass of Analytic Functions with Negative Coefficients.

AMER, A. A. & DARUS, M. 2011. On some properties for new generalized derivative operator. *Jordan Journal of Mathematics and Statistics (JJMS),* 4**,** 91-101.

BRANNAN, D. A. & TAHA, T. 1988. On some classes of bi-univalent functions. *Mathematical Analysis and Its Applications.* Elsevier.

DUREN, P. L. 2001. *Univalent functions*, Springer Science & Business Media.

JUMA, A. & AZIZ, F. S. 2012. Applying Ruscheweyh derivative on two sub-classes of bi-univalent functions. *Inter. J. of Basic & Appl. Sci,* 12**,** 68-74.

MOTAMEDNEZHAD, A., SALEHIAN, S. & MAGESH, N. 2022. Coefficient estimates for subclass of m-fold symmetric bi-univalent functions. *Kragujevac Journal of Mathematics,* 46**,** 395-406.

SABIR, P. O. 2024. Some remarks for subclasses of bi-univalent functions defined by Ruscheweyh derivative operator. *Filomat,* 38**,** 1255-1264.
SHMELLA, E. K. & AMER, A. A. 2024a. Estimation of the Bounds of Univalent Functional of Coefficients Apply the Subordination Method.
SHMELLA, E. K. & AMER, A. A. 2024b. Some properties of Differential Subordination for the Subordination Class with the Generalized Derivative Operator. مجلة جامعة بني وليد للعلوم الإنسانية والتطبيقية, 390-400.