Necessary conditions for the generalized derivative operator in classes of univalent functions

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Abstract:

The main aim of our investigation is to obtain necessary conditions for the generalized derivative operator to belong to the classes $S_\alpha(\theta)$ and $S_\alpha(\theta)$ **.** In addition, we obtain the distortion theorems for the classes $S_\alpha(\theta)$ and $S_\alpha(\theta)$ of our main results.

1- Introduction:

Complex Function Theory (CFT) emerged in the eighteenth century as a mathematical field. Within (CFT), functions are characterized as complex-valued and analytic in a specific domain. Furthermore, if a function's derivative exists at z_0 in its domain, it is said to be analytic (regular or holomorphic). Given that these functions are analytic, they have Taylor series developments in their domain, and can thus be expressed

in a specific series form with centers at z_0, and can be written as

$$f\,(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots.$$

Several researchers, including Euler, Gauss, Riemann, Cauchy, and others, were interested

in this branch because it has a wide range of applications in mathematics and science, and also has many interesting properties that real-valued functions do not have. Furthermore, the class of the starlike and convex functions; they are also univalent and it is worth noting that O.M. Reade [11] studied the concept of classes after Kaplan [12] introduced them. Several exciting subclasses of the univalent function class have been studied previously from various perspectives. For example, Owa et al. [14], Kowalczyk and Le- Bomba [15] (see Gao and Zhou [16]. Also, the standard books [7,9] can be looked into for several interesting geometric properties of these classes.

Identified an open unit disk $U = \{z \in \mathbb{C}; |z| < 1\}$, with respect to the complex plane in which $\mathcal{A}$ refers to the class of functions f given by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \qquad (1)$$

This is analytic in U satisfying the usual normalization conditions given by

$$\grave{f}(0) = 1 + f(0) = 1\,.$$

The Hadamard product (also known as convolution) for two analytic functions f as is in equation (1) and

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k \quad , \qquad (z \in U).$$

is provided by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k \ ,$$

And by using this product, the authors in [8] have recently introduced a new generalized derivative operator given by:

Definition 1.1 :

The shifted factorial $(c)_k$ can be defined as:

$$(c)_k = c(c+1)\ldots\ldots(c+k-1) \ \ if \quad k \in \mathbb{N} = \{1,2,3,\ldots\}, \quad c \in \mathbb{C} - \{0\}.$$

and

$$(c)_k = 1 \ \ if \ k = 0.$$

Definition 1.2 :

The $(c)_k$ can be expressed in terms of the Gamma function as :

$$(c)_k = \frac{\Gamma(c+k)}{\Gamma(c)} \ \ ,(n \in \mathbb{N}).$$

In order to derive the generalized derivative operator [8], we define the analytic function

$$\phi^m(\lambda_1,\lambda_2,l)(z) = z + \sum_{k=2}^{\infty} \frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}(1+\lambda_2(k-1))^m} z^k, \qquad (1)$$

Where $m \in \mathbb{N}_0 = \{0,1,2,....\}$ and λ_2 $,\lambda_1, l \in \mathbb{R}$ such that $\lambda_2 \geq \lambda_1 \geq 0\, , l \geq 0$.

Definition 1.3

For $f \in \mathcal{A}$ the operator $I^m(\lambda_1,\lambda_2,l,n)$ is defined by $I^m(\lambda_1,\lambda_2,l,n)\colon \mathcal{A} \longrightarrow \mathcal{A}$

$$I^m(\lambda_1,\lambda_2,l,n)f(z) = \phi^m(\lambda_1,\lambda_2,l)(z) * R^n f(z) \quad ,(z \in U) \qquad (2)$$

Where $m \in \mathbb{N}_0 = \{0,1,2,....\}$ and $\lambda_2 \geq \lambda_1 \geq 0\, , l \geq 0$, and $R^n f(z)$ denotes the Ruscheweyh derivative operator [18] ,and given by

$$R^n f(z) = z + \sum_{k=2}^{\infty} c(n,k) a_k z^k = z + \sum_{k=2}^{\infty} \frac{(n+1)_{k-1}}{(k-1)!} a_k z^k\, , (n \in \mathbb{N}_0, z \in U),$$

If f is given by (1) , then we easily find from the equality (3) that

$$I^m(\lambda_1,\lambda_2,l,n)f(z) = z + \sum_{k=2}^{\infty} \frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}\big(1+\lambda_2(k-1)\big)^m} c(n,k) a_k z^k,$$

Where $n,m \in \mathbb{N}_0 = \{0,1,2,...\}$, $\lambda_2 \geq \lambda_1 \geq 0\, , l \geq 0$, $(c)_k = \frac{\Gamma(c+k)}{\Gamma(c)}$.

Special cases of this operator includes:

- The Ruscheweyh derivative operator [18] in the cases :

  $$I^1(\lambda_1,0,l,n) \equiv I^1(\lambda_1,0,0,n) \equiv I^1(0,0,l,n) \equiv I^0(0,\lambda_2,0,n) \equiv I^0(0,0,0,n) \equiv I^{m+1}(0,0,l,n) \equiv I^{m+1}(0,0,0,n) \equiv R^n.$$

- The Salagean derivative operator [19] :

  $$I^{m+1}(1,0,0,0) \equiv D^n.$$

- The generalized Ruscheweyh derivative operator[20] :

  $$I^2(\lambda_1,0,0,n) \equiv R_\lambda^n.$$

- The generalized Salagean derivative operator introduced by [21] :

  $$I^{m+1}(\lambda_1,0,0,0) \equiv D_\beta^n.$$

- The generalized Al-Shaqsi and Darus derivative operator [20] :

  $$I^{m+1}(\lambda_1,0,0,n) \equiv D_{\lambda,\beta}^n\ .$$

- The Al-Abbadi and Darus generalized derivative operator [22] :

  $$I^m(\lambda_1,\lambda_2,0,n) \equiv \mu_{\lambda_1,\lambda_2}^{n,m}\ .$$

- And finally the catas derivative operator [23] :

$I^m(\lambda_1, 0, l, n) \equiv I^m(\lambda_1, \beta, l).$

Using simple computation one obtains the next result

$$(l+1)I^{m+1}(\lambda_1, \lambda_{2,} l, n)f(z) = (1+l-\lambda_1)[I^m(\lambda_1, \lambda_{2,} l, n) * \varphi^1(\lambda_1, \lambda_2, l)(z)]f(z) + \lambda_1 z\left[I^m\left(\lambda_1, \lambda_{2,} l, n\right) * \varphi^1(\lambda_1, \lambda_2, l)(z)\right]', \qquad (*)$$

where $(z \in U)$ and $\varphi^1(\lambda_1, \lambda_2, l)(z)$ analytic function given by

$$\phi^1(\lambda_1, \lambda_2, l)(z) = z + \sum_{k=2}^{\infty} \frac{1}{(1+\lambda_2(k-1))} z^k \ .$$

Definition 1.4 A function f belonging to $\mathcal{A}$ is said to be in the class $S^*(\lambda_1, \lambda_2, l, n, \alpha)$ in U if it satisfies

$$Re\left(\frac{z(I^m(\lambda_1, \lambda_2, l, n)f(z))'}{I^m(\lambda_1, \lambda_2, l, n)f(z)}\right) > \alpha \quad , \quad (z \in U),$$

for some $\alpha(0 \le \alpha < 1)$.

Definition 1.5 A function f belonging to $\mathcal{A}$ is said to be in the class $C(\lambda_1, \lambda_2, l, n, \alpha)$ in U if it satisfies

$$Re\left(\frac{(z\left(I^m(\lambda_1, \lambda_2, l, n)f(z)\right)')'}{(I^m(\lambda_1, \lambda_2, l, n)f(z))'} + 1\right) > \alpha, \quad (z \in U),$$

for some $\alpha(0 \le \alpha < 1)$.

We note that

$$f(z) \in C(\lambda_1, \lambda_2, l, n, \alpha) \Leftrightarrow zf'(z) \in S^*(\lambda_1, \lambda_2, l, n, \alpha). \qquad (3)$$

The authors [17] found the class of analytic functions in open unit disk normalized by the conditions $f(0) = 0, f'(0) = 1,$ where $\alpha(0 < \alpha < 1),$ are real numbers

Now, for some real number $\alpha(0 < \alpha < 1),$ we know that

$$\left|\frac{I^m(\lambda_1, \lambda_2, l, n) f(z)}{z(I^m(\lambda_1, \lambda_2, l, n) f(z))'} - \frac{1}{2\alpha}\right| < \frac{1}{2\alpha} \Leftrightarrow \text{Re}\left\{\frac{z(I^m(\lambda_1, \lambda_2, l, n) f(z))'}{I^m(\lambda_1, \lambda_2, l, n) f(z)}\right\} > \alpha.$$

Then, it is easy to show that

$$\left|\frac{I^m(\lambda_1, \lambda_2, l, n) f(z)}{z(I^m(\lambda_1, \lambda_2, l, n) f(z))'} - \frac{1}{2\alpha}\right| < \frac{1}{2|\alpha|} \Leftrightarrow \text{Re}\left\{\frac{1}{\alpha} \frac{z(I^m(\lambda_1, \lambda_2, l, n) f(z))'}{I^m(\lambda_1, \lambda_2, l, n) f(z)}\right\} > 1.$$

for some complex number α such that $|\alpha-\frac{1}{2}|<\frac{1}{2}.$

From the above reason, we consider the new subclass of $\mathcal{A}$ involving our generalized derivative operator $I^m(\lambda_1,\lambda_2,l,n)f(z)$.

If $f\in\mathcal{A}$ satisfies the following inequality

$$\Re\left\{\frac{1}{\alpha}\frac{z(I^m(\lambda_1,\lambda_2,l,n)f(z))'}{I^m(\lambda_1,\lambda_2,l,n)f(z)}\right\}>1 \qquad (z\in U),$$

for some complex number α such that $|\alpha-\frac{1}{2}|<\frac{1}{2}$, then we say that $f\in S_\alpha(\lambda_1,\lambda_2,l,n).$

We note that the function $f\in S_\alpha(0,0,0,0)$ is spirallike in U which implies that f is univalent in

Similarly, if $f\in\mathcal{A}$ satisfies the following inequality

$$\Re\left\{\frac{1}{\alpha}\left(1+\frac{z(I^m(\lambda_1,\lambda_2,l,n)f(z))''}{(I^m(\lambda_1,\lambda_2,l,n)f(z))'}\right)\right\}>1 \qquad (z\in U),$$

for some complex number α such that $|\alpha-\frac{1}{2}|<\frac{1}{2}$, then we say that $f\in C_\alpha(\lambda_1,\lambda_2,l,n).$

From the relation (3), it is clear that

$$f\in C_\alpha(\lambda_1,\lambda_2,L,n)\Leftrightarrow zf'(z)\in S_\alpha(\lambda_1,\lambda_2,L,n), \qquad (4)$$

for some complex number α such that $|\alpha-\frac{1}{2}|<\frac{1}{2}.$

2 Necessary conditions for $S_\alpha(\theta)$ and $C_\alpha(\theta)$.

As a result, the following new a subclass of $\mathcal{A}$ is defined

$$A(\theta)=\{I^m(\lambda_1,\lambda_2,l,n)f(z)\in A: I^m(\lambda_1,\lambda_2,l,n)f(z)=z+\sum_{k=2}^{\infty}\gamma_n|a_k|e^{i((k-1)\theta+\pi)}z^k\},$$

$$S_\alpha(\theta)=A(\theta)\cap S_\alpha(\lambda_1,\lambda_2,l,n) \text{ and } C_\alpha(\theta)=A(\theta)\cap C_\alpha(\lambda_1,\lambda_2,l,n),$$

for some $\theta(0\le\theta<2\pi)$, where $\gamma_n=\frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}(1+\lambda_2(k-1))^m}c(n,k).$

This definition was used to describe our problem via the following necessary and sufficient criteria were satisfied for $S_\alpha(\theta)$ and $C_\alpha(\theta)$.

Now,we discuss the necessary conditions for $S_\alpha(\theta)$ and $C_\alpha(\theta)$.

Theorem 2.1

If a function $f \in S_\alpha(\theta)$ for some complex number $\alpha \in \mathbb{C} \setminus \{0\}$ and $\theta (0 \le \theta < 2\pi)$,

then

$$\sum_{k=2}^{\infty} \frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}(1+\lambda_2(k-1))^m} c(n,k)(k\,\mathrm{Re}(\alpha) - |\alpha|^2)\,|a_k| \le \mathrm{Re}(\alpha) - |\alpha|^2 .$$

Proof:

By the definition of $S_\alpha(\theta)$, we can assume that

$$\mathrm{Re}\left\{ \frac{1}{\alpha} \frac{z(I^m(\lambda_1,\lambda_2,l,n) f(z))'}{I^m(\lambda_1,\lambda_2,l,n) f(z)} \right\}$$

$$= \mathrm{Re}\left\{ \frac{1}{\alpha} \frac{z + \sum_{k=2}^{\infty} k \frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}(1+\lambda_2(k-1))^m} c(n,k)\,|a_k|\, e^{i((k-1)\theta+\pi)} z^k}{z + \sum_{k=2}^{\infty} \frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}(1+\lambda_2(k-1))^m} c(n,k)\,|a_k|\, e^{i((k-1)\theta+\pi)} z^k} \right\} > 1.$$

Setting $\alpha = |\alpha| e^{i\arg(\alpha)}$ and $z = |z| e^{i\theta}$, we obtain the following inequality

$$\mathrm{Re}\left\{ \frac{1}{|\alpha| e^{i\arg(\alpha)}} \frac{1 - \sum_{k=2}^{\infty} k \frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}(1+\lambda_2(k-1))^m} c(n,k)\,|a_k|\,|z|^{k-1}}{1 - \sum_{k=2}^{\infty} \frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}(1+\lambda_2(k-1))^m} c(n,k)\,|a_k|\,|z|^{k-1}} \right\} > 1,$$

In a similar direction, we have

$$\mathrm{Re}\left\{ \frac{\cos(\arg(\alpha))}{|\alpha|} \frac{1 - \Sigma_{k=2}^{\infty} k \frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}(1+\lambda_2(k-1))^m} c(n,k)\,|a_k|\,|z|^{k-1}}{1 - \sum_{k=2}^{\infty} \frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}(1+\lambda_2(k-1))^m} c(n,k)\,|a_k|\,|z|^{k-1}} \right\} > 1.$$

Put $|z| \to 1$, then we get

$$\cos(\arg(\alpha)) \left(1 - \sum_{k=2}^{\infty} k \frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}(1+\lambda_2(k-1))^m} c(n,k)\,|a_k| \right)$$

$$\ge |\alpha| \left(1 - \Sigma_{k=2}^{\infty} \frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}(1+\lambda_2(k-1))^m} c(n,k)\,|a_k| \right)$$

that is ,that

$$\sum_{k=2}^{\infty}\frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}(1+\lambda_2(k-1))^m}c(n,k)\left(k\,\mathrm{Re}(\alpha)-|\alpha|^2\right)|a_k|\le \mathrm{Re}(\alpha)-|\alpha|^2 .$$

Corollary 2.1 If a function $f\in S_\alpha(\theta)$ and $\lambda_2=0, m=1$,then

$$\sum_{k=2}^{\infty}(k\,\mathrm{Re}(\alpha)-|\alpha|^2)|a_k|\le \mathrm{Re}(\alpha)-|\alpha|^2,$$

for some complex number α such that $|\alpha-\frac{1}{2}|<\frac{1}{2}$, then $f\in S_\alpha$,see [6].

Similarly,we obtain the coefficient inequality for $f\in C_\alpha(\theta)$.

Theorem 2.2

If a function $f\in C_\alpha(\theta)$ for some complex number $\alpha\in\mathbb{C}\setminus\{0\}$ and $\theta(0\le\theta<2\pi)$, then

$$\sum_{k=2}^{\infty}k\frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}(1+\lambda_2(k-1))^m}c(n,k)\left(k\,\mathrm{Re}(\alpha)-|\alpha|^2\right)|a_k|\le \mathrm{Re}(\alpha)-|\alpha|^2 .$$

Corollary 2.2

If a function $f\in C_\alpha(\theta)$ and $\lambda_2=0, m=1$,then

$$\sum_{k=2}^{\infty}k(k\,\mathrm{Re}(\alpha)-|\alpha|^2)|a_k|\le \mathrm{Re}(\alpha)-|\alpha|^2,$$

for some complex number α such that $|\alpha-\frac{1}{2}|<\frac{1}{2}$, then $f\in C_\alpha$,see [6].

Next, applying Theorem 2.1 and Theorem 2.2, we consider the distortion theorems for the classes $S_\alpha(\theta)$ and $C_\alpha(\theta)$.

3 The Distortion Theorems for $S_\alpha(\theta)$ and $C_\alpha(\theta)$.

Theorem 3.1 If a function $f\in S_\alpha(\theta)$ for $\theta(0\le\theta<2\pi)$, then

$$r-\delta_j-\frac{\mathrm{Re}(\alpha)-|\alpha|^2-\mu_j}{(j+1)\,\mathrm{Re}(\alpha)-|\alpha|^2}r^{j+1}\le|I^m f(z)|\le r+\delta_j+\frac{\mathrm{Re}(\alpha)-|\alpha|^2-\mu_j}{(j+1)\,\mathrm{Re}(\alpha)-|\alpha|^2}r^{j+1},$$

where

$I^m f(z)=I^m(\lambda_1,\lambda_2,l,n)f(z)$,

$$\delta_j = \begin{cases} 0 & if \quad (j=1), \\ \sum_{k=2}^{j} \frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}(1+\lambda_2(k-1))^m} c(n,k)\,|a_k|\,r^k & if \quad (j=2,3,4,\cdots), \end{cases}$$

and

$$\mu_j = \begin{cases} 0 & if \quad (j=1), \\ \sum_{k=2}^{j} \frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}(1+\lambda_2(k-1))^m} c(n,k)(k\operatorname{Re}(\alpha)-|\alpha|^2)\,|a_k| & if \quad (j=2,3,4,\cdots). \end{cases}$$

Proof:

By Theorem 2.2, it follows that, for $I^m f(z) \in S_\alpha(\theta)$,

$$\sum_{k=2}^{j} \frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}(1+\lambda_2(k-1))^m} c(n,k)(k\operatorname{Re}(\alpha)-|\alpha|^2)\,|a_k| + \{(j+1)\operatorname{Re}(\alpha)-|\alpha|^2\} \sum_{k=j+1}^{\infty} |a_k|$$

$$\leq \sum_{k=2}^{\infty} \frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}(1+\lambda_2(k-1))^m} c(n,k)(k\operatorname{Re}(\alpha)-|\alpha|^2)\,|a_k| \leq \operatorname{Re}(\alpha)-|\alpha|^2,$$

or

$$\sum_{k=j+1}^{\infty} \frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}(1+\lambda_2(k-1))^m} c(n,k)\,|a_k| \leq \frac{\operatorname{Re}(\alpha)-|\alpha|^2-\mu_j}{(j+1)\operatorname{Re}(\alpha)-|\alpha|^2} \quad (j=1,2,3,\cdots).$$

We see that

$$I^m f(z) \leq r + \sum_{k=2}^{\infty} \frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}(1+\lambda_2(k-1))^m} c(n,k)\,|a_k|\,r^k$$

$$= r + \sum_{k=2}^{j} \frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}(1+\lambda_2(k-1))^m} c(n,k)\,|a_k|\,r^k$$

$$+ \sum_{k=j+1}^{\infty} \frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}(1+\lambda_2(k-1))^m} c(n,k)\,|a_k|\,r^k$$

$$\leq r + \delta_j + \frac{\operatorname{Re}(\alpha)-|\alpha|^2-\mu_j}{(j+1)\operatorname{Re}(\alpha)-|\alpha|^2} r^{j+1},$$

Already, we have

$$I^m f(z) \geq r - \sum_{k=2}^{\infty} \frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}(1+\lambda_2(k-1))^m} c(n,k)|a_k| r^k$$

$$= r - \sum_{k=2}^{j} \frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}(1+\lambda_2(k-1))^m} c(n,k)|a_k| r^k$$

$$- \sum_{k=j+1}^{\infty} \frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}(1+\lambda_2(k-1))^m} c(n,k)|a_k| r^k$$

$$\geq r - \delta_j - \frac{\text{Re}(\alpha) - |\alpha|^2 - \mu_j}{(j+1)\text{Re}(\alpha) - |\alpha|^2} r^{j+1},$$

which is the desired result.

Setting $j = 1$ in Theorem 3.1,we get

Corollary 3.1 If a function $f \in S_\alpha(\theta)$ for $\theta(0 \leq \theta < 2\pi)$, then

$$r - \frac{\text{Re}(\alpha) - |\alpha|^2}{2\text{Re}(\alpha) - |\alpha|^2} r^2 \leq |I^m f(z)| \leq r + \frac{\text{Re}(\alpha) - |\alpha|^2}{2\text{Re}(\alpha) - |\alpha|^2} r^2,$$

with equality for

$$I^m f(z) = z - \frac{\text{Re}(\alpha) - |\alpha|^2}{2\text{Re}(\alpha) - |\alpha|^2} e^{i\theta} z^2, (z = \pm r e^{-i\theta}).$$

Using the same technique, we can discuss the similar theorem for $C_\alpha(\theta)$.

Theorem 3.2 If a function $f \in C_\alpha(\theta)$ for $\theta(0 \leq \theta < 2\pi)$, then

$$r - \delta_j - \frac{\text{Re}(\alpha) - |\alpha|^2 - \mu_j^*}{(j+1)\{(j+1)\text{Re}(\alpha) - |\alpha|^2\}} r^{j+1} \leq |I^m f(z)| \leq r + \delta_j + \frac{\text{Re}(\alpha) - |\alpha|^2 - \mu_j^*}{(j+1)\{(j+1)\text{Re}(\alpha) - |\alpha|^2\}} r^{j+1},$$

where

$$\delta_j = \begin{cases} 0 & if \quad (j=1), \\ \sum_{k=2}^{j} \frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}(1+\lambda_2(k-1))^m} c(n,k)|a_k| r^k & if \quad (j = 2,3,4,\cdots), \end{cases}$$

So, as a result,

$$\mu_j^* = \begin{cases} 0 & if \quad (j=1), \\ \sum_{k=2}^{j} k \frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}(1+\lambda_2(k-1))^m} c(n,k)(k\,\text{Re}(\alpha) - |\alpha|^2)|a_k| & if \quad (j = 2,3,4,\cdots). \end{cases}$$

Setting $j=1$ in Theorem 3.4,we get

Corollary 3.2

If a function $f \in C_{\alpha}(\theta)$ for $\theta(0 \le \theta < 2\pi)$, then

$$r-\frac{\operatorname{Re}(\alpha)-|\alpha|^2}{2(2\operatorname{Re}(\alpha)-|\alpha|^2)}r^2 \le |I^m f(z)| \le r+\frac{\operatorname{Re}(\alpha)-|\alpha|^2}{2(2\operatorname{Re}(\alpha)-|\alpha|^2)}r^2,$$

with equality for

$$I^m f(z)=z-\frac{\operatorname{Re}(\alpha)-|\alpha|^2}{2(2\operatorname{Re}(\alpha)-|\alpha|^2)}e^{i\theta}z^2,(z=\pm re^{-i\theta}).$$

Moreover, we also derive the following results.

Theorem 3.3

If a function $f \in S_{\alpha}(\theta)$ for $\theta(0 \le \theta < 2\pi),(|z|=r)$, then

$$1-\delta_j^*-\frac{(j+1)\{\operatorname{Re}(\alpha)-|\alpha|^2-\mu_j\}}{(j+1)\operatorname{Re}(\alpha)-|\alpha|^2}r^j \le |(I^m f(z))'| \le 1+\delta_j^*+\frac{(j+1)\{\operatorname{Re}(\alpha)-|\alpha|^2-\mu_j\}}{(j+1)\operatorname{Re}(\alpha)-|\alpha|^2}r^{j+1},$$

where

$$\delta_j^*=\begin{cases} 0 & if \quad (j=1),\\ \sum_{k=2}^{j} k\frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}(1+\lambda_2(k-1))^m}c(n,k)|a_k|r^{k-1} & if \quad (j=2,3,4,\cdots),\end{cases}$$

and

$$\mu_j=\begin{cases} 0 & if \quad (j=1),\\ \sum_{k=2}^{j} \frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}(1+\lambda_2(k-1))^m}c(n,k)(k\operatorname{Re}(\alpha)-|\alpha|^2)|a_k| & if \quad (j=2,3,4,\cdots).\end{cases}$$

Proof:

Note that

$$\sum_{k=2}^{j}\frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}(1+\lambda_2(k-1))^m}c(n,k)(k\operatorname{Re}(\alpha)-|\alpha|^2)|a_k|+\frac{(j+1)\operatorname{Re}(\alpha)-|\alpha|^2}{j+1}\sum_{k=j+1}^{\infty}k|a_k|$$

$$\le\sum_{k=2}^{\infty}\frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}(1+\lambda_2(k-1))^m}c(n,k)(k\operatorname{Re}(\alpha)-|\alpha|^2)|a_k| \le \operatorname{Re}(\alpha)-|\alpha|^2,$$

or

$$\sum_{k=j+1}^{\infty} k\frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}(1+\lambda_2(k-1))^m}c(n,k)|a_k| \le \frac{(j+1)(\mathrm{Re}(\alpha)-|\alpha|^2-\mu_j)}{(j+1)\mathrm{Re}(\alpha)-|\alpha|^2} \quad (j=1,2,3,\cdots).$$

We see that

$$(I^m f(z))' \le 1+\sum_{k=2}^{\infty} k\frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}(1+\lambda_2(k-1))^m}c(n,k)|a_k|r^{k-1}$$

$$=1+\sum_{k=2}^{j} k\frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}(1+\lambda_2(k-1))^m}c(n,k)|a_k|r^{k-1}$$

$$+\sum_{k=j+1}^{\infty} k\frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}(1+\lambda_2(k-1))^m}c(n,k)|a_k|r^{k-1}$$

$$\le 1+\delta_j^* + \frac{(j+1)\{\mathrm{Re}(\alpha)-|\alpha|^2-\mu_j\}}{(j+1)\mathrm{Re}(\alpha)-|\alpha|^2}r^{j+1},$$

and

$$(I^m f(z))' \ge 1-\sum_{k=2}^{\infty} k\frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}(1+\lambda_2(k-1))^m}c(n,k)|a_k|r^{k-1}$$

$$=1-\sum_{k=2}^{j} k\frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}(1+\lambda_2(k-1))^m}c(n,k)|a_k|r^{k-1}$$

$$-\sum_{k=j+1}^{\infty} k\frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}(1+\lambda_2(k-1))^m}c(n,k)|a_k|r^{k-1}$$

$$\ge 1-\delta_j^* - \frac{(j+1)\{\mathrm{Re}(\alpha)-|\alpha|^2-\mu_j\}}{(j+1)\mathrm{Re}(\alpha)-|\alpha|^2}r^{j+1}.$$

Setting $j=1$ in Theorem 3.3,we have

Corollary 3.3

If a function $f \in S_\alpha(\theta)$ for $\theta(0\le\theta<2\pi)$, then

$$1-\frac{2(\mathrm{Re}(\alpha)-|\alpha|^2)}{2(2\mathrm{Re}(\alpha)-|\alpha|^2)}r \le |(I^m f(z))'| \le 1+\frac{2(\mathrm{Re}(\alpha)-|\alpha|^2)}{2(2\mathrm{Re}(\alpha)-|\alpha|^2)}r,$$

with equality for

$$I^m f(z) = z-\frac{\mathrm{Re}(\alpha)-|\alpha|^2}{2\mathrm{Re}(\alpha)-|\alpha|^2}e^{i\theta}z^2, (z=\pm re^{-i\theta}).$$

Using the same method, we can get the similar theorem for $C_\alpha(\theta)$.

Theorem 3.4

If a function $f \in C_\alpha(\theta)$ for $\theta(0 \le \theta < 2\pi)$, then

$$1-\delta_j^* - \frac{\operatorname{Re}(\alpha)-|\alpha|^2-\mu_j^*}{(j+1)\{(j+1)\operatorname{Re}(\alpha)-|\alpha|^2\}} r^j \le |(I^m f(z))'| \le 1+\delta_j^* + \frac{\operatorname{Re}(\alpha)-|\alpha|^2-\mu_j^*}{(j+1)\{(j+1)\operatorname{Re}(\alpha)-|\alpha|^2\}} r^j,$$

where

$$\delta_j^* = \begin{cases} 0 & if \quad (j=1), \\ \sum_{k=2}^{j} k \frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}(1+\lambda_2(k-1))^m} c(n,k)|a_k| r^k & if \quad (j=2,3,4,\cdots), \end{cases}$$

and

$$\mu_j^* = \begin{cases} 0 & if \quad (j=1), \\ \sum_{k=2}^{j} k \frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}(1+\lambda_2(k-1))^m} c(n,k)(k\operatorname{Re}(\alpha)-|\alpha|^2)|a_k| & if \quad (j=2,3,4,\cdots). \end{cases}$$

Setting $j=1$ in Theorem 3.4,we get

Corollary 3.4

If a function $f \in C_\alpha(\theta)$ for $\theta(0 \le \theta < 2\pi)$, then

$$1-\frac{\operatorname{Re}(\alpha)-|\alpha|^2}{2\operatorname{Re}(\alpha)-|\alpha|^2} r \le |(I^m f(z))'| \le 1+\frac{\operatorname{Re}(\alpha)-|\alpha|^2}{2\operatorname{Re}(\alpha)-|\alpha|^2} r,$$

with equality for

$$I^m f(z) = z - \frac{\operatorname{Re}(\alpha)-|\alpha|^2}{2(2\operatorname{Re}(\alpha)-|\alpha|^2)} e^{i\theta} z^2, (z = \pm r e^{-i\theta}).$$

Recently, a number of researchers have studied various several problems
for univalent functions involving different operators and different classes, and
many other work on analytic functions related to the generalized derivative operator can be read in [1,2,3,4,5,8,10,13,15,17].

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