Stability theory for systems of ordinary differential equations

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Abstract

This study focuses on stability theory for systems of differential equations, concentrating in particular on systems of first-order ordinary differential equations. For this types of system, definitions of the appropriate stability concepts are provided and considering some important methods of establishing stability for equilibrium points ,whereby one can obtain sufficient conditions for these concepts to apply. For doing this, investigations in detail are made to the methods of Lyapunov for ordinary differential equation systems.

المستخلص

تم التركيز في هده الدراسة على نظرية الاستقرار لمنضومة المعادلات التفاضلية, وقد ثم التركيز على المعادلات التفاضلية من الرتبة الأولى. حيث قدمنا تعاريف مناسبة لمفاهيم الاستقرار و درسنا بعض الطرق المهمة لتقييم الاستقراريه لنقاط الاتزان التي بواسطتها نستطيع تحقيق الشروط الكافية لهده المفاهيم للتطبيق.كما تدارسنا في هدا العمل في تفاصيل طرق ليابونوف لمنظومة المعادلات التفاضلية العادية.

1 Introduction

Systems of first-order differential equations are ubiquitous throughout applied mathematics. The general system formulation can be written as

$$x' = \Delta(t, x),$$

where Δ is an all-encompassing quantity taken to represent whatever dependence may be present on the time t and the N-dimensional state variable x.

There are many interesting and important questions pertaining to such a system. Of course, two of the most significant are those of existence and uniqueness of solutions. However, supposing we are in a setting in which the existence and uniqueness of solutions is known, there is particular question that is perhaps more important than any other within the setting of any application. This question concerns a vital quantity of physical relevance, the stability the solutions: is an arbitrary solution stable, in the sense of perturbations off this solution not growing in time?

Today various methods exist for investigating the stability of solution of linear and nonlinear systems of differential equations. In this paper, we are concerned with those techniques first developed by A. Liapunov. It is based on the concept that the potential energy of a conservative dynamical system has a relative minimum at a stable equilibrium point. This second method has been recognized to be very general and powerful in the qualitative theory of differential equations for the
reason that questions of stability can be addressed without actually having the solution of the system.

2 Basic Concepts And Definitions

we introduce the concepts of stability and asymptotic stability for solutions of a differential equation and consider some methods that may be used to prove stability.

To introduce the concepts, consider the simple scalar equation

$$y'(t) = ay(t). \qquad (1.1)$$

The solution is, of course, $y(t) = y_0 e^{at}$, where $y_0 = y(0)$. In particular, $y(t) \equiv 0$ is a solution.

What happens if we start at some point other that 0?

If $a < 0$, then every solution approaches 0 as $t \to \infty$. We say that the zero solution is(globally) asymptotically stable.

and the direction field of the equation, i.e., the arrows have the same slope as the solution that passes through the tail point.
If we take $a = 0$ in (1.1), the solutions are all constant. This does have some relevance to stability: if we start near the zero solution, we stay near the zero solution. In this case, we say the zero solution is *stable*, (but not asymptotically stable).
Finally, if $a > 0$ in (1.1), every nonzero solution goes to infinity as t goes to infinity.
In this case, no matter how close to zero we start, the solution is eventually far away from zero. We say the zero solution is *unstable*.

2.1 Characteristic Roots:

To solve the linear system with constant coefficients

$$\begin{aligned} x' &= a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ x_2' &= a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \\ &\dots\dots \\ &\dots\dots \\ x_n' &= a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n \end{aligned} \qquad (1.2)$$

recall that we try for a solution of the form

$$\begin{aligned} x_1 &= A_1 e^{\lambda t} \\ x_2 &= A_2 e^{\lambda t} \\ &\dots \\ &\dots \\ x_n &= A_n e^{\lambda t} \end{aligned}$$

This leads to the characteristic equation

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & \dots\dots a_{1n} \\ a_{21} & a_{22} & \dots\dots a_{2n} \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{nn}-\lambda \end{vmatrix} = 0$$

This is a polynomial equation of degree n in with real coefficients. The variety of possible roots of such an equation is well known, and the types of terms that appear in the general solution of (1.2) are of the form

$$e^{\lambda t} p(t), \qquad (1.3$$

Where λ is a characteristic root (possibly imaginary) and $p(t)$ is a polynomial in t [1].

2.2 Definitions of stability for ODE systems

The precise form of ODE system on which we shall focus is

$$x' = f(t, x); \qquad (1.4)$$

with $f : [0,\infty) \times D \to R^n$ piecewise continuous in t and locally Lipschitz in x, where D is a domain containing the origin. When necessary, the derivative here shall be regarded as representing the right-hand derivative.

We now define the relevant stability concepts for the system (1.4),

Definition : A solution $x = X(t)$ of (1.4) is said to be:

- stable if, given any $\varepsilon > 0$ and any $t_0 \geq 0$, there exists a $\delta = \delta(\varepsilon, t_0)$ such that

$$|x(t_0)-X(t_0)|<\delta \Rightarrow |x(t)-X(t)|<\varepsilon, \forall t \ge t_0 \ge 0; \qquad (1.5)$$

for any solution $x(t)$ of (1.4),

- uniformly stable if, for every ε > 0, there exits $\delta = \delta(\varepsilon)$, independent of t_0, such that (1.5) is satisfied for all $t_0 \ge 0$,
- unstable if it is not stable,
- asymptotically stable if it is stable and for any $t_0 \ge 0$ there exists a positive constant $c = c(t_0)$ such that

$$|x(t_0 - X(t_0))| < c \Rightarrow x(t) - X(t) \to 0 \text{ as } t \to \infty \ ;$$

For a any solution $x(t)$ of (1.4),

- uniformly asymptotically stable if it is uniformly stable and there exists a positive constant c, independent of t_0, such that, for every η > 0; there exists $T = T(\eta) > 0$ such that, for all $t_0 \ge 0$

$$|x(t_0)-X(t_0)| < c \Rightarrow |x(t)-X(t)| < \eta, \forall t \ge t_0 + T(\eta);$$

for any solution $x(t)$ of (1.4),

- globally uniformly asymptotically stable if it is uniformly stable with $\delta(\varepsilon)$ satisfying $\lim_{\varepsilon\to\infty} \delta(\varepsilon) = \infty$, and, for all positive η and c, there exists $T = T(\eta, c)$ > 0 such that, for all $t_0 \ge 0$

$$|x(t_0)-X(t_0)| < c \Rightarrow |x(t)-X(t)| < \eta, \forall t \ge t_0 + T(\eta, c);$$

for any solution $x(t)$ of (1.4) [2].

3 Stability theory of solutions for systems of differential equations

3.1 Stability of Linear Systems

Consider the linear homogeneous system

$$x' = Ax(t); \qquad (3.1)$$

where A is an constant $n \times n$ matrix. The system may be real or complex. We know, of course, that the solution is

$$x(t) = e^{At}x_0, \qquad x(0) = x_0.$$

Thus, the origin is an equilibrium point for this system. we can characterize the stability of this equilibrium point.

Define

$$p = a + d = trace(A), \ q = ad - bc = \det(A).$$

Then, the eigenvalues are the roots of the characteristic polynomial

$$\det(A - \lambda I) = (\lambda - a)(\lambda - b) - bc = \lambda^2 - (a + d)\lambda + ad - bc = \lambda^2 - p\lambda + q$$

Let λ_1, λ_2 be the two eigenvalues. Then we must have

$$\lambda_1 + \lambda_2 = p, \qquad \lambda_1\lambda_2 = q. \qquad (3.2)$$

Let $\Delta = p^2 - 4q$ denote the discriminant of the characteristic polynomial. Then, the two roots can be written as

$$\lambda_{1,2} = \frac{p \pm \sqrt{\Delta}}{2}.$$

We now discuss several cases.

• If Δ < 0 and p = 0, we have pure imaginary eigenvalues. The equilibrium is a center.

• If $\Delta < 0$ and $p > 0$, we have complex eigenvalues with positive real parts. The equilibrium is a spiral point, which is unstable.
• If $\Delta < 0$ and $p < 0$, we have complex eigenvalues with negative real parts. The equilibrium is a spiral point, which is asymptotically stable.
• If $\Delta > 0$, then the two eigenvalues are real and distinct. Using the relation (3.2), we see that:
– If $q < 0$, then λ_1, λ_2 have the opposite sign, and we have a saddle point which is unstable.
– If $q > 0$, then λ_1, λ_2 have the same sign, and we have a proper node. If $p > 0$, they are positive, so the node is unstable. Otherwise, if $p < 0$, they are negative and we have a stable node.
• Finally, if $\Delta = 0$, the eigenvalues are repeated, and we have either a proper node or an improper node. If $p > 0$, it is unstable, and if $p < 0$, it is asymptotically stable [3].
See graph below for an illustration.

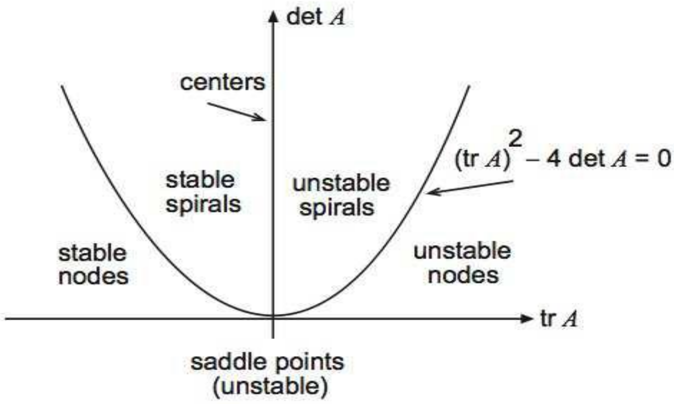

Figure1: Stability diagram

Theorem 3.1. Let A be an $n \times n$ matrix and let the spectrum of A (i.e., the eigenvalues of A) be denoted by $\sigma(A)$ and consider the linear system of differential equations (3.1).
1. If $\text{Re}(\sigma(A)) \le 0$ and all the eigenvalues of A with real part zero are simple, then 0 is a stable fixed point for (3.1).
2. If $\text{Re}(\sigma(A)) < 0$, then 0 is a globally asymptotically stable solution of (3.1).
3. If there is an eigenvalue of A with positive real part, then 0 is unstable [4].

3.1.1 Summary of Stabilities and types of critical points for linear systems

For the 2 × 2 system

$$x' = Ax$$

we see that $x = (0, 0)$ is the only critical point if A is invertible.
In a more general setting: the system

$$x' = Ax - b$$

would have a critical point at $x' = A^{-1}b$.

The type and stability of the critical point is solely determined by the eigenvalues of A .
Summary of types and stabilities of critical points:

$\lambda_{1,2}$	Eigenvalues	type of C.P.	Stability
Real	$\lambda_1 \cdot \lambda_2 < 0$	saddle point	Unstable
Real	$\lambda_1 > 0$, $\lambda_2 > 0$, $\lambda_1 \neq \lambda_2$	node (source)	Unstable
Real	$\lambda_1 < 0, \lambda_2 < 0, \lambda_1 \neq \lambda_2$	node (sink)	A.S.(asymptotically stable)
Real	$\lambda_1 = \lambda_2$, 2 eigenvectors	proper node/star point	A.S. if $\lambda_1 < 0$, unstable if $\lambda_1 > 0$
Real	$\lambda_1 = \lambda_2$, one eigenvector	improper node	A.S. if $\lambda_1 < 0$, unstable if $\lambda_1 > 0$
Imaginary	$\lambda_{1,2} = \pm i\beta$	Center	stable but not asymptotically
Complex	$\lambda_{1,2} = \alpha \pm i\beta$	spiral point	A.S. if $\alpha < 0$, unstable if $\alpha > 0$

As long stability is concerned, the sole factor is the sign of the real part of the eigenvalues [5].
If any of eigenvalue shall have a positive real part, the it is unstable.

3.2 Autonomous systems and their critical points

Let $x(t), y(t)$ be the unknowns, we consider the system

$$\begin{cases} x'(t) = F(x, y) \\ y'(t) = G(x, y) \end{cases} \qquad \begin{cases} x(t_0) = x_0, \\ y(t_0) = y_0. \end{cases}$$

for some functions $f(x, y), G(x, y)$ that do not depend on t. Such a system is called autonomous. Typical examples are in population dynamics, which we will see in our examples.
Using matrix-vector form, one could also write an autonomous system as

$$\vec{x}'(t) = \vec{F}\left(\vec{x}\right), \qquad \vec{x}(t_0) = \vec{x}_0 .$$

A critical point is a point such that the right hand-side is 0, i.e.,

$$f(x, y) = 0 \quad , \qquad G(x, y) = 0$$

or in the vector notation

$$\vec{F}\left(\vec{x}\right) = 0.$$

Note that, since now the functions are non-linear, there could be multiple critical points.
Finding zeros for a nonlinear vector-valued function could be a non-trivial task.
We first go through some examples on how to find the critical points.
Example: Find all critical points for

$$\begin{cases} x'(t) = -(x-y)(1-x-y) \\ y'(t) = x(2+y) \end{cases}$$

Answer. We see that the right hand-sides are already in factorized form, which makes our task easier. We must now require

$$x = y, \quad \text{or} \quad x - y = 1$$

And

$$x = 0 \quad \text{or} \quad y = -2.$$

We see that we have 4 combinations.

$$(1) \quad \begin{cases} x = y \\ x = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases}$$

$$(2) \quad \begin{cases} x = y \\ y = -2 \end{cases} \Rightarrow \begin{cases} x = -2 \\ y = -2 \end{cases}$$

$$(3) \quad \begin{cases} x + y = 1 \\ x = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 1 \end{cases}$$

$$(4) \quad \begin{cases} x + y = 1 \\ y = -2 \end{cases} \Rightarrow \begin{cases} x = 3 \\ y = -2 \end{cases}$$

General strategy.
(1) Factorize the right hand as much as you can.
(2) Find the conditions for each equation.
(3) Make all combinations and solve.

3.3 Stability of fixed points of nonlinear equations

In this section, we consider some important methods of establishing stability for equilibrium points (a.k.a., fixed points) of nonlinear differential equations. This approach is detailed in the reference [8].

3.3.1 Stability by linearization

Let $f : R^n \to R^n$ be *a* C^1 map and suppose that *p* is a point such that *f*(*p*) = 0, i.e., *p* is a fixed point for the differential equation $x'(t) = f(x(t))$.

The linear part of f at p, denoted *Df*(*p*), is the matrix of partial derivatives at *p*: For $x \in R^n, f(x) \in R^n$, so we can write

$$f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \\ . \\ . \\ . \\ f_n(x) \end{bmatrix}.$$

The functions fi are called the component functions of *f*. We define

$$Df(p)=\begin{bmatrix} \frac{\partial f_1}{\partial x_1}(p) & \frac{\partial f_1}{\partial x_2}(p) & \cdots\frac{\partial f_1}{\partial x_n}(p) \\ \frac{\partial f_2}{\partial x_1}(p) & \frac{\partial f_2}{\partial x_2}(p) & \cdots\frac{\partial f_2}{\partial x_n}(p) \\ . & . & . \\ . & . & . \\ . & . & . \\ \frac{\partial f_n}{\partial x_1}(p) & \frac{\partial f_n}{\partial x_2}(p) & \cdots\frac{\partial f_n}{\partial x_n}(p) \end{bmatrix}.$$

Since f is C^1, Taylor's theorem for functions of several variables says that

$$f(x)=Df(p)(x-p)+g(x).$$

(we've used $f(p)=0$), where g is a function that is small near p in the sense that

$$\lim_{x\to p}\frac{|g(x)|}{|x-p|}=0.$$

An important result is that p is an asymptotically stable equilibrium point if $\mathrm{Re}\,(\sigma(Df(p)))<0$, i.e., if the origin is asymptotically stable for the linear system $y'=Df(p)y$. We prove this result in the next theorem, which is a bit more general. Note that by introducing the change of coordinates $y=x-p$, we may assume without loss of generality that the fixed point is at the origin.

Theorem3. 3.1. Let A be an n × n real constant matrix with Re(σ (A)) < 0. Let g be a function with values in R^n, defined on an open subset U of $R\times R^n$ that contains $[0,\infty)\times B_r(0)$, for some r > 0. We assume that g is continuous and locally Lipschitz with respect to the second variable, g(t ,0) = 0 for all t, and

$$\lim_{x\to 0}\frac{|g(t,x)|}{|x|}=0, \qquad \text{uniformly for } t\in[0,\infty) \qquad (3.3)$$

Under these conditions, the origin is an asymptotically stable fixed point of the nonlinear system

$$x'(t)=Ax(t)+g(t,x(t)) \qquad (3.4)$$

Proof. Since Re(σ A)) < 0, we can find K >0 and σ < 0 such that

$$\left\|e^{At}\right\|\le Ke^{\sigma t}, \quad t\ge 0.$$

Let x(t) be a solution of the system (3.4) defined on some interval (a; b) containing 0 and let $x_0=x(0)$. Using the integrating factor e^{-At} on the equation $x'(t)-Ax(t)=g(t,x(t))$, we conclude that

$$x(t)=e^{At}x_0+\int_0^t e^{A(t-s)}g(s,x(s))ds.$$

Taking norms, we get

$$|x(t)|\le K|x_0|e^{\sigma t}+\int_0^t Ke^{\sigma(t-s)}|g(s,x(s))|ds, \qquad t\in[0,b).$$

We may multiply this equation through by $e^{-\sigma t}$ to get

$$e^{-\sigma t}|x(t)|\le K|x_0|+\int_0^t Ke^{-\sigma s}|g(s,x(s))|ds, \quad t\in[0,b). \quad (3.5)$$

Choose η > 0 su±ciently small that $\eta K<-\sigma$. By (3.3), there is some δ > 0 such that δ < r and

$$\frac{g(t,x)}{|x|} \le \eta$$

for all (t , x) such that $t \in [0,\infty)$ and $0 < |x| \le \delta.$. To put it another way, we have

$$(t,x) \in [0,\infty) \times \overline{B}_\delta(0) \Rightarrow |g(t,x)| \le \eta |x|. \qquad (3.6)$$

Define $\delta' = \delta/(2K+1) < \delta$. Suppose that $|x_0| < \delta'$, and let x(t) be the maximally defined solution of (3.4) with $x(0) = x_0$. Denote the interval of definition of this solution by (a , b). We claim that $b = \infty$ and that $x(t) \to 0$ as $t \to \infty$. If we show this, the proof that the origin is asymptotically stable with be complete.

Suppose that $[0,c] \subseteq (a,b)$ is an interval such that $|x(t)| \le \delta$ for all $t \in [0,c]$. We can find some such interval by the continuity of x. By (3.6), we have $|g(t,x(t))| \le \eta |x(t)|$ for $t \in [0,c]$. If we substitute this in (3.5), we get

$$e^{-\sigma t}|x(t)| \le K|x_0| + \int_0^t e^{-\sigma s} K\eta |x(s)| ds, \qquad t \in [0,c]$$

We now apply Gronwall's inequality, with (in the notation of that theorem) $f_1(t) = e^{-\sigma t}|x(t)|, f_2 = K|x_0|$ and $p(s) = K\eta$. As a result, we get the inequality

$$e^{-\sigma t}|x(t)| \le K|x_0| + \int_0^t K^2 |x_0| \eta e^{K\eta(t-s)} ds, \qquad t \in [0,c].$$

Evaluating the integral, we get

$$e^{-\sigma t}|x(t)| \le K|x_0| + K^2|x_0|\eta \frac{1}{K\eta}\left|e^{K\eta t} - 1\right| = K|x_0|e^{K\eta t}.$$

Since $|x_0| < \delta'$, $K|x_0| < \delta$, and so $e^{-\sigma t}|x(t)| \le \delta e^{K\eta t}$. Thus, we have

$$|x(t)| \le \delta e^{(\sigma + K\eta)t}, \qquad t \in [0,c] \qquad (3.7)$$

where $\sigma + K\eta < 0$.

Of course, we've only shown that this inequality holds under the initial assumption that $|x(t)| \le \delta$ on [0 *,* *c*]. Let *S* be the set of numbers *c* > 0 such that [0 *,* *c*] is contained in the interval (*a* *,* *b*) and $|x(t)| \le \delta$ for $t \in [0,c]$. As we observed above, *S* is not empty. Let *s* be the supermom of *S*.

If 0 < *v* < *s*, then *v* is not an upper bound for *S* and there must be some element *c* of *S* such that *v* < *c* ≤ *s*. But then $v \in [0,c]$ and by the definition of *s*, $|x(t)| \le \delta$ for $t \in [0,c]$.

So, we must have $|x(v)| \le \delta$. This argument shows that $|x(t)| \le \delta$ on $[0,s)$.

We claim that $s = \infty$. To prove this, suppose (for a contradiction) that *s* is finite.

Since *b* is an upper bound for *S*, we must have $s \le b$. Suppose that *b* = *s*. Then, the right hand endpoint of the interval of existence of *x* is finite. It follows that *x*(*t*) must leave the compact set $\overline{B}_r(0)$ as $t \to s$. But this does not happen because $|x(t)| \le \delta < r$ for $t \in [0,s)$.

So, we must have *s* < *b*.

This means that *s* is in the domain of *x*. By continuity, we must have $|x(s)| \le \delta$ But this means that $|x(t)| \le \delta$ on [0 *,* *s*], and so we may apply the inequality (4.3.26) on the interval [0 *,* *s*] to conclude that $|x(s)| \le \delta e^{(\sigma + K\eta)t} < \delta$

. But then, by continuity, we can find some $\varepsilon > 0$ such that $s < s + \varepsilon < b$ and $|x(t)| \le \delta$ on $[s, s+\varepsilon]$. It follows that x is defined on $[0, s+\varepsilon]$ and $|x(t)| \le \delta$ on $[0, s+\varepsilon]$. But this means that $s + \varepsilon \in S$, which contradicts the definition of s as the supermom of S. This contradiction shows that $s = \infty$.

Now, let $u > 0$ be arbitrary. Since $u < s$, $|x(t)|$ is bounded by δ on $[0, u]$. But then we can apply (3.7) on the interval $[0, u]$ to conclude that $|x(u)| \le \delta e^{(\sigma+K\eta)u}$. Since u was arbitrary, we conclude that

$$|x(t)| \le \delta e^{(\sigma+K\eta)T}, \quad t \in [0,\infty),$$

and so $|x(t)| \to 0$ as t goes to infinity. This completes the proof.

We remark that if the origin is stable but not asymptotically stable for the linear system $x' = Ax$, the origin may *not* be stable for the nonlinear system (3.4)|the nonlinear part is significant in this case [6],[7].

3.4 Lyapunov stability theory

Based on the foregoing definitions, it is natural to seek sufficient conditions that ensure each relevant notion of stability of system (1.4). We now present in detail some of the theory in this area, beginning with methods for some simplified special cases of (1.4) before building up to the main ideas for the fully general system.

Note1: As is standard in the literature, the theory will be presented for the case of an equilibrium solution at x = 0. However, a coordinate transformation of the form $x(t) \to x(t) + y(t)$ allows the theory to be applied about any solution y(t) of the original system (1.4).

3.4.1 Nonlinear autonomous systems

A natural next step in the analysis of systems of the form (1.4) is to continue to require that the right-hand side have no explicit time-dependence, but to allow f to be a more general nonlinear functional

$$x' = f(x) \qquad (4.1)$$

This was the first major problem addressed by Aleksandr Mikhailovich Lyapunov in his Doctoral Thesis of 1892 . Lyapunov considered the legitimacy of expanding the nonlinear function f as a Taylor Series about the equilibrium $x = 0$

$$f(x) \approx f(0) + \frac{\partial f}{\partial x}(0)x + O(x^2)$$

$$\approx \frac{\partial f}{\partial x}(0)x + O(x^2).$$

If the initial state $x(0) = x_0$ is chosen close enough to 0, then x will be `small' for some time interval extending from zero. Intuitively, this suggests that we should be able to neglect the higher-order terms, and approximate our nonlinear system (4.1) by the linear system

$$x' = Ax, \text{ Where } A = \frac{\partial f}{\partial x}(0).$$

Lyapunov, in what is now known as Lyapunov's Indirect Method or Lyapunov's First Method, made precise when this approximation can be used to determine the stability properties of the system (4.1).

Theorem3.4.1 (Lyapunov's Indirect Method). Let $x = 0$ be an equilibrium of the nonlinear system (4.1), where

$f : D \to R^n$ is continuously differentiable. Let

$$A = \frac{\partial f}{\partial x}(0).$$

Then:

- $x = 0$ is asymptotically stable if $R(\lambda_i) < 0$ for all eigenvalues λ_i of A,

• $x=0$ is unstable if $R(\lambda_i)>0$ for some eigenvalue λ_i of A.
The proof of this theorem is rather involved, and will not be given here.
Lyapunov's Indirect Method allows one to test for stability of a nonlinear system by calculating the eigenvalues of the Jacobian matrix at the equilibrium point. However, if all eigenvalues have $R(\lambda_i)\leq 0$ but some $R(\lambda_i)=0$ then linearization fails to determine the stability of the equilibrium, and higher-order terms in the series expansion of f become significant.

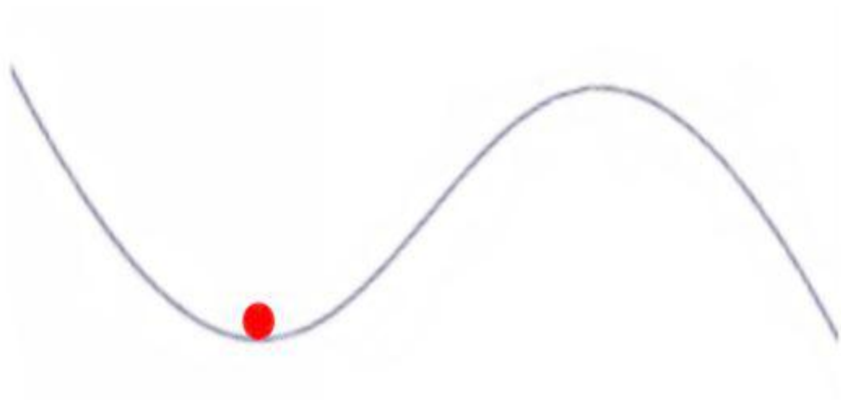

Figure 2: The ball is at a local minimum of its gravitational potential energy, and so is in a locally stable state.

3.4.2. Nonautonomous systems

Lyapunov also developed what has come to be known as Lyapunov's Direct Method, or Lyapunov's Second Method. This method allows us to extend our consideration to the more general nonautonomous system, where the right-hand side is allowed to depend explicitly on t. The exact form to be considered is that stated in (1.4)

$$x' = f(t,x),$$

with $f:[0,\infty)\times D\to R^n$ piecewise continuous in t and locally Lipschitz in x, where D is a domain containing the origin.
The central idea of Lyapunov's Direct Method is to generate a function V , now commonly known as a Lyapunov function, that is essentially a generalization of a physical energy function. In physics, a well-known consequence of the Second Law of Thermodynamics is the Principle of Minimum Total Potential Energy , which states that:
An object within any dissipative physical system will move and deform so as to minimize its total potential energy.
It was known, therefore, that any state of an object in a physical system could only be stable to small perturbations if it was a local minimum of the body's potential energy. An example of this is illustrated in Figure 1.
Lyapunov's key realization was that this concept could be generalized to the more general system (1.4) by introducing a function that plays the role of the potential energy. Specifically, in conjunction with our intuitive picture of dissipative forces, the Principle of Minimum Total Potential Energy tells us that, in a stable region, the total energy of the system decreases towards some local minimum along all paths of evolution. This suggests, by analogy, that a candidate Lyapunov function should decrease along all system trajectories in a neighborhood of our equilibrium, towards a local minimum at the equilibrium. If this occurs, and our analogy holds as we expect, then this should prove that the equilibrium in question is stable. This is the essence of the following theorem.

Theorem3.4. 2 (Lyapunov's Direct Method). Let $x=0$ be an equilibrium of the system (1.4) and $U\subset D$ be a domain containing $x=0$. Suppose that there exists a continuous function $V:[0,\infty)\times R^N\to R$, such that, with the time-derivative along the system trajectories defined as $V'(t,x(t)) = \limsup_{h\to 0+\frac{1}{h}} \{V(t+h, x(t+h)) - V(t,x(t))\}, V$

satisfies:

i. $V(t,0) = 0, \forall t \geq 0,$

ii. $V(t,x) \geq W_1(x), \forall t \geq 0, \forall x \in U$, for some continuous positive definite function W1 on U,

iii. $V'(t,x) \leq 0, \forall t \geq t_0, \forall x \in U$.

Then the equilibrium $x = 0$ is stable.

Proof. Fix an arbitrary $t_0 \geq 0$. Since V satisfies conditions i and ii above and U is a domain containing 0, there exist a constant r > 0 and a strictly increasing continuous function α satisfying $\alpha(0) = 0$ such that $B_r \subset U$ and

$$\alpha(|x|) \leq V(t,x), \forall t \geq 0, \forall x \in B_r. \qquad (4.2)$$

Now let ε > 0 be arbitrary and fixed. In order to prove stability of $x = 0$, we need to show that there exists a $\delta = \delta(\varepsilon, t_0)$ such that (1.5) holds. Begin by setting $\varepsilon_1 = \min\{\varepsilon, r\}$ and choosing δ > 0 such that

$$\sup_{|x|\leq\delta} V(t_0, x) =: \beta(t_0, \delta) < \alpha(\varepsilon_1).$$

This can always be done because $\alpha(\varepsilon_1)$ > 0 and $\beta(t_0, \delta) \to 0$ as $\delta \to 0$.

Suppose now that $|x(t_0)| < \delta$, and let τ be the smallest time t at which $|x(t) \geq \varepsilon_1|$. This is well defined because x is a continuous function. Then, by definition

$$|x(t)| < \varepsilon_1, \forall t \in [t_0, \tau), \text{ and } |x(\tau)| = \varepsilon_1. \quad (4.3)$$

Therefore, as $\varepsilon_1 \leq r,$, property iii gives tha

$$\frac{d}{dt} V(t, x(t)) \leq 0, \forall t \in [t_0, \tau),$$

whence

$$V(\tau, x(\tau)) \leq V(t_0, x(t_0)) < \alpha(\varepsilon_1)$$

But, by (4.2) and (4.3), we also have

$$V(\tau, x(\tau)) \geq \alpha(|x(\tau)|) = \alpha(\varepsilon_1),$$

giving a contradiction.

Therefore, no such τ can exist. Thus, as $\varepsilon_1 \leq \varepsilon$, this shows that

$$|x(t_0)| < \delta \Rightarrow |x(t)| < \varepsilon, \forall t \geq t_0,$$

with ε > 0 and $t_0 \geq 0$ arbitrary, which proves the result.

The conditions in the above theorem may be strengthened in a number of ways in order to give sufficient conditions for the various different forms of stability possible for the system (1.4). We now state those extensions

to cover the important cases of uniform stability and uniform asymptotic stability. We omit the proofs since they proceed by arguments similar to the proof of Theorem 3. However, for the interested reader, these proofs, along with other modifications of Lyapunov's Direct Method, are well-documented in the literature.

Theorem3.4. 3 (Lyapunov Theorem for Uniform Stability). Let $x = 0$ be an equilibrium of the system (1.4) and $U \subset D$ be a domain containing $x = 0$. Suppose that there exists a continuous function

$V:[0,\infty)\times R^N \to R,$ such that, with the time-derivative along the system trajectories defined as $V'(t,x(t)) = \limsup_{h\to 0+\frac{1}{h}} \{V(t+h,x(t+h)) - V(t,x(t))\}, V$ satisfies:

i. $W_1(x)\le V(t,x)\le W_2(x), \forall t\ge 0, \forall x\in U,$, for some continuous positive definite functions W1, W2 on U,

ii. $V'(t,x)\le 0, \forall t\ge 0, \forall x\in U$.

Then the equilibrium $x=0$ is uniformly stable.

Note 2: Note that W1(0) = W2(0) = 0, so that V (t , 0) = 0 is still required of any Lyapunov function.

Theorem3.4. 4 (Lyapunov Theorem for Uniform Asymptotic Stability). Suppose that all the assumptions of Theorem 3 are satisfied with ii strengthened to:

$$V'(t,x)\le -W_3(x), \forall t\ge 0, \forall x\in U,$$

for some continuous positive definite function W3 on U. Then the equilibrium $x=0$ is uniformly asymptotically stable.

If further $U=D=R^n$ and $W_1(x)\to\infty$ as $|x|\to\infty$, then the solution $x=0$ is globally uniformly asymptotically stable.

Theorems3.4. 2, 3.4.3 and 3.4.4, along with the many other closely related results, provide a clear path for proving stability properties of ODE systems by means of finding a Lyapunov function. Moreover, several converse Lyapunov theorems have been developed, guaranteeing that, under mild assumptions, if certain stability properties of a system hold. This means that seeking a Lyapunov function inspired by knowledge of the system being studied is very often a good method of approach. However, Lyapunov's Direct Method, are well-documented in the literature, for instance in [2], [9], and [10].

5. Conclusion:

The stability of fixed points of a system of linear differential equations of first order can be analyzed using the eigenvalues of the corresponding matrix if all eigenvalues are negative real numbers or complex numbers with negative real parts then the point is a stable .

One of the key ideas in stability theory is that the qualitative behavior of an orbit under perturbations can be analyzed using the linearization of the system near the orbit.

The most significant idea within this study was the thinking behind Lyapunov's Direct Method: the idea that we can often prove stability of a solution of an ODE system through the use of an auxiliary Lyapunov function.

References

[1] Eunice Umoh Umanah, Stability theory of differential equations via Liapunov's techniques, Atlanta University Center. (1979). ETD Collection for AUC Robert W. Woodruff Library. Paper 689.

[2] Eoin Devane, Stability theory for systems of differential equations with applications to power control in wireless networks, Cambridge Centre for Analysis, January 14, 2013

[3] Introduction to Ordinary and Partial Differential Equations, Wen Shen, Spring 2015.

[4] A.C.King, J.Billingham and S.R. Otto, Differential Equations Linear, Nonlinear, Ordinary, Partial, CAMBRIDGE UNIVERSITY PRESS.

[5] William E. Boyce, Richard C. Diprima, Elementary Differential Equations, John Wiley & Sons,Inc 2004.

[6] D.A. Sanchez, Ordinay Differential Equations and Stability Theory: An Introduction, W.H. Freeman and Company, 1968.

[7] F. Brauer, J.A. Nohel, Qualitative Theory of Ordinary Differential Equtions, Dover,1969.

[8] Gerald Teschl, Ordinary Differential Equations and Dynamical Systems, American Mathematical Society (AMS),2012

[9] Richard Haberman, APPLIED PARTIAL DIFFERENTIAL EQUATIONS With Fourier Series and Boundary Value Problems, PEARSON EDUCATION, New Jersey2004

[10] Jeffrey R. Chasnov, Introduction to Differential Equations, The Hong Kong University of Science and Technology , 2009