Good starting points for iteration of Newton's function.

أحسن نقط بداية لتكرار دالة نيوتن

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Abstract

In this paper we want to describe the algorithm to find all good starting points for iteration of $N_p(z)$ to find all the roots of $p(z)$ by quick and easy way. So in this paper we have proved that all critical points go to the roots under iteration of Newton's function then it will be the good starting points for iteration of $N_p(z)$.

Keywords: Newton's function, Starting points, Immediate Basins.

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الملخص

في هذه الورقة البحثية نريد ان نصف طريقة ايجاد جميع نقط البداية لتكرار دالة نيوتن $N_p(z)$ وذلك لايجاد جذور كثيرة الحدود $p(z)$ بطريقة سهلة وسريعة. ولذلك في هذه الورقة البحثية اثبتنا انه كل النقاط الحرجة تقترب من جذور كثيرة الحدود تحت عملية تكرار دالة نيوتن وبالتالى تكون النقاط الحرجه افضل نقاط بداية لتكرار دالة نيوتن $N_p(z)$.

مفتاح الكلمات: دالة نيوتن، نقط بداية التكرار ، الاحواض الفورية للجذور

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Introduction

It is a fundamental problem to find all roots of a complex polynomial p. Newton's method is one of the most widely known numerical algorithms for finding the roots of complex polynomials , starting with an arbitrary starting point $z \in \mathbb{C}$ iterate the Newton's map $z \to N(z) = z - \frac{p(z)}{p'(z)}$ until the iterate sufficiently close to a root. Newton's method is widely used due to its simplicity and because of its efficient in practice near a simple root, the convergence is quadratic (the number of valid digits double in every iteration step). (Gilbert, 1991).

Definition.

The basin of attraction for a fixed point z_0 is

$$A(z_0) = \{z \in \mathbb{C}: R^n(z) \to z_0 \ \ as \ \ n \to \infty\}.$$

And the basin of attraction for a periodic cycle $p = p_1, p_2, \dots, p_n$ is

$$A(p) = \{z \in \mathbb{C} : \ R^{in}(z) \to p_k \ , \qquad for \ some \ k \in \{1,2,\dots,n\}, \\ n = 1,2,3,\dots, as \ i \to \infty\}$$

(Peitgen, 1989).

Definition

Let z_0 be a (super)attracting fixed point then the connected component of $A(z_0)$ containing z_0 is called the immediate basin of attraction of z_0 and denoted by $A^*(z_0)$. (Carlesonand Gamelin, 1992).

Newton's method and complex dynamical system

Let

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_{d-1} z^{d-1} + a_d z^d \qquad (1)$$

be a polynomial with real coefficients and only real (and simple) zeros x_k, $1 \le k \le d$.
Let

$$N(z) = z - \frac{p(z)}{p'(z)} \qquad (2)$$

be the Newton's function associated with p.
So $p: \mathbb{C} \to \mathbb{C}$, the rational function $N: \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, and the fixed points of N given By

$$\frac{p(z)}{p'(z)} = 0 \ \Rightarrow p(z) = 0.$$

Thus the fixed points of N are the zeros of p together with ∞.
Differentiating we find that

$$N'(z) = \frac{p(z)p''(z)}{(p'(z))^2} \qquad (3)$$

And $N'(x_k) = 0$ because $p(x_k) = 0$, $1 \le k \le d$.
This means that the zeros of p are a super-attracting fixed points of N. If $|z|$ is large ,$N(z) \sim z\left(1 - \frac{1}{d}\right)$ where$d$ is the degree of p, so ∞ is a repelling fixed point of N.
Now we have

$$N'(z) = \frac{p(z)p''(z)}{(p'(z))^2}$$

If $N'(x_k) = 0 \Longrightarrow p(z) = 0 \ \ at \ \ x_1, x_2, \dots, x_d \in \mathbb{R} \ or \ p''(z) = 0$
At $\zeta_1, \zeta_2, \dots, \zeta_{d-1} \in \mathbb{R}$. So $N(z)$ has $2d - 2$ critical points.

Basic properties of Newton's function

Throughout this section p will denote a polynomial from $\mathbb{R}$ to $\mathbb{R}$. We will start with the following assumption
(i) (a) If $p(x) = 0$, then $p'(x) \neq 0$.

(b) If $p'(x) = 0$, then $p''(x) \neq 0$.

As we have said before

$N(x) = x - \frac{p(x)}{p'(x)}$ (4)

denotes the Newton function associated with p, the fundamental property of N is that, it transforms the problem of finding roots of p, into a problem of finding attracting fixed points of N. Note $N' = \frac{pp''}{(p')^2}$.

(ii) $p(x) = 0$ if and only if $N(x) = x$. Moreover, if $p(\alpha) = 0$, then $N'(\alpha) = 0$ so $N^k(x) \to \alpha$ for all x near α. (Alan, 1991).

Definition

If $c_1 < c_2$ are consecutive roots of $p'(x)$, then the interval (c_1, c_2) is called a band for N.

Definition

If $p'(x)$ has the largest (respectively, smallest) root c (respectively, b), then the interval (c, ∞) (respectively, $(-\infty, b)$) is called an extreme band for N (Fram, 1944).

(iii) If (c_1, c_2) is a band for N that contains a root of $p(x)$, then

$$\lim_{x \to c_1^+} N(x) = +\infty, \qquad \lim_{x \to c_2^-} N(x) = -\infty .$$

Proof(iii)

We know that $\frac{p(x)}{p'(x)} < 0$ in (c_1, x_k) then $N(x) > x$ in (c_1, x_k) thus $\lim_{x \to c_1^+} N(x) = +\infty$, and $\frac{p(x)}{p'(x)} > 0$ in (x_k, c_2) then $N(x) < x$ in (x_k, c_2) thus $\lim_{x \to c_2^-} N(x) = -\infty$.

Immediate Basins of Newton's Function.

In this section we want to show that each immediate basin of fixed points of Newton Function for a real polynomial is simply connected.

Theorem

Let

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_{d-1} z^{d-1} + a_d z^d.$$

be a polynomial with real coefficient and only real (and simple) zeros x_k , $1 \leq k \leq d$ and let (c_1, c_2) be a band contains x_k and ζ_k (zero of $p''(x)$ which is a free critical point), so the interval $[x_k, \zeta_k] \subset (c_1, c_2)$. Then the interval between the fixed point x_k and free critical point ζ_k is mapped by N into itself, i.e. $N[x_k, \zeta_k] \subset [x_k, \zeta_k]$.

point ζ_k is mapped by N into itself, i.e. $N[x_k, \zeta_k] \subset [x_k, \zeta_k]$.

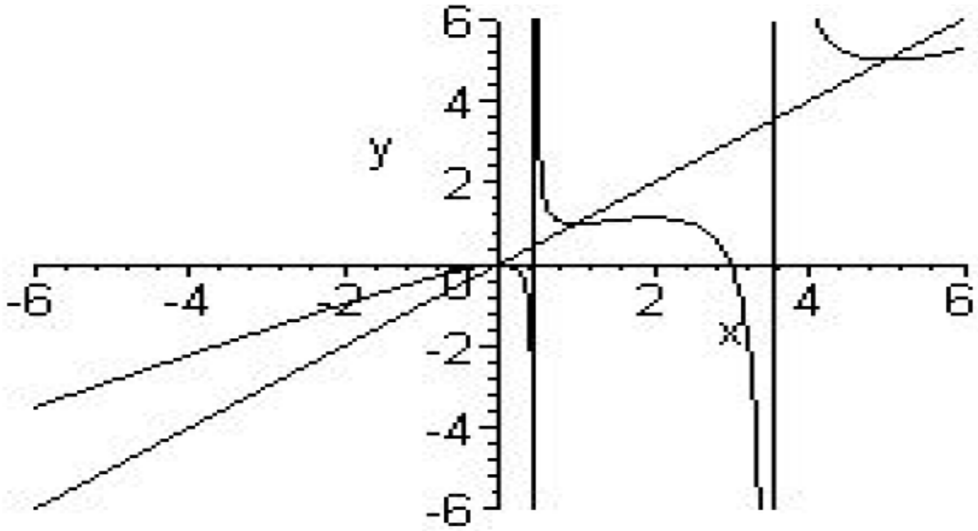

Figure 1: Newton function for the polynomial $p(x) = (x-1)x(x-5)$

Proof

We know that $N(x_k) = x_k,\ N'(x_k) = 0,\ N'(\zeta_k) = 0$ and we have to consider two cases which are $x_k < \zeta_k$ or $x_k > \zeta_k$. So let start with the first case where $x_k < \zeta_k$ in the previous section we have proved that

$$\lim_{x \to c_1^+} N(x) = +\infty, \qquad \lim_{x \to c_2^-} N(x) = -\infty .$$

Then $N(x) > x_k$ in (c_1, x_k) it follows that

$$N(x) > x_k \quad in \ (x_k, \zeta_k) \tag{5}$$

and since $N(x) < \zeta_k$ in (ζ_k, c_2) then

$$N(x) > \zeta_k \quad in \ (x_k, \zeta_k) \tag{6}$$

from (5) and (6) we have

$$N[x_k, \zeta_k] \subset [x_k, \zeta_k] \tag{7}$$

Now the second case where $x_k > \zeta_k$, again since $N(x) < x_k$ in (x_k, c_2) then

$$N(x) < x_k \quad in \ (\zeta_k, x_k) \tag{8}$$

and since$N(x) > \zeta_k$ in (c_1, ζ_k) then

$$N(x) > \zeta_k \quad in \ (\zeta_k, x_k) \tag{9}$$

from (8) and (9) we have

$$N[\zeta_k, x_k] \subset [\zeta_k, x_k] \quad (10)$$

It also follows that $\zeta_k \in A^*(x_k)$, where $A^*(x_k)$, is the immediate basin of attraction of x_k.

Theorem

Let $p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_{d-1} z^{d-1} + a_d z^d$ be a polynomial with distinct and real roots x_k , $1 \le k \le d$, and let N be the Newton's function associated with $p(z)$.
So that x_k, $1 \le k \le d$ will be supperattracting fixed points of Newton's function of $p(z)$. Then the immediate basin of each x_k , $1 \le k \le d$, is simply connected.

Proof

We know that x_k, $1 \le k \le d$, are super-attracting fixed points and ζ_k, $2 \le k \le d-1$, are free critical points of $N_p(z)$. Let V be a small neighborhood of x_k contained in $A^*(x_k)$, i.e.
$V = \Delta(x_k, \epsilon) = \{z: |z - x_k| < \epsilon\}$, where $\epsilon > 0$. Such that $N(\zeta_k) \notin V$, so that V is simply connected, then the connectivity number of $V = m = 1$. Let $V_1 =$ component of $N^{-1}(V)$ contains x_k, and since $N(\zeta_k) \notin V$ then $\zeta_k \notin V_1$ then V_1 contains one critical point which is x_k, so the topological degree $k = 2$ then, $N: V_1 \overset{2:1}{\rightarrow} V$ is a proper mapping, so by applying Riemann Hurwitz Formula, (Steinmetz, 1993).

$$m - 2 = k(n-2) + r$$

where n is the connectivity number of V_1, and $r = 1$ (number of critical points). Then

$$1 - 2 = 2(n-2) + 1 \Rightarrow n = 1$$

Then V_1 is simply connected as well. After i backward iterations, we get V_i =component of $N^{-i}(V)$ which contains x_k. Since N is mapping $[x_k, \zeta_k]$ into itself, then we can find i in such a way that $N(\zeta_k) \in V_i$, but $\zeta_k \notin V_i$, so until i backward iterations there are no changes in the numbers of connectivity and the numbers of critical points on V and V_i, it follows that V_i is simply connected too. Now let us take $i+1$ backward iterations, which is the important step in this proof, we get $V_{i+1} =$ component of $N^{-(i+1)}(V)$ contains x_k.
Since $N^{-1}[x_k, N(\zeta_k)] = [x_k'] \cup [\zeta_k, x_k']$,
Then $\zeta_k, x_k' \in V_{i+1}$, where x_k' is pre-image of x_k. Since V_{i+1} contains two critical points which are x_k, ζ_k, then the topological degree $k = 3$, so $N: V_{i+1} \overset{3:1}{\rightarrow} V_i$ is proper mapping, and the connectivity number of $V_i = m = 1$, then by applying Riemann Hurwitz Formula once more

$$m - 2 = k(n-2) + r$$

Where n the connectivity number of V_{i+1}, and $r = 2$, then

$$1 - 2 = 3(n-2) + 2 \Rightarrow n = 1$$

Therefore V_{i+1} is simply connected as well. Since there are no more critical points, then by the same argument we can prove that $V_{i-2}, V_{i-3}, \ldots$ are also simply connected. It follows that the whole immediate basin of x_k is simply connected

Conclusion

For most starting points of most polynomials, it does not take to long to find good approximations of a root. But there are problems, through there are starting points which never converge to a root under the Newton's map (for example all the points on the boundaries of the basins of all the roots). Therefore we suggest good in other ward convenient starting points to find all real roots of a polynomials of degree$d$ with d distinct real roots, these good starting points are ζ_k , $2 \leq k \leq d-1$, where ζ_k are zeros of $p''(z)$ and that is because we have proved that $N^n(\zeta_k) \longrightarrow x_k$, then ζ_k are good starting points for iteration of N. So we can describe the algorithm to find all good starting points for iteration of $N_p(z)$ to find all the roots of p(z) , as the following,

1) If d is even then we find the zeros of $p^{(d-2)}(z)$, where $p^{(d-2)}(z)$ is the derivative of $p(z)$, $d-2$ times. So they are two good starting points for Iteration of $N_p(z)$ to find the two roots in the middle which are $x_{\frac{d}{2}}$, $x_{\frac{d}{2}+1}$. Then we find the zeros of $p^{(d-4)}(z)$, so we get another two starting points for iteration of $N_p(z)$ to find another two roots which are $x_{\frac{d}{2}-1}$, $x_{\frac{d}{2}+2}$. We keep going by the same steps until we find the zeros of $p''(z)$ then we have two starting points for iteration of N_p to find the two roots x_2, x_{d-1}. Now for the last two roots which are x_1, x_d we can start the iteration of $N_p(z)$ with any point $x < x_1$ to get the root x_1 and with any point $x > x_d$ to get the root x_d
2) If d is odd then we find the zero of $p^{(d-1)}(z)$, so it will be a good starting point for iteration of $N_p(z)$ to find the middle root which is $x_{\frac{d+1}{2}}$. Then we find the zeros of $p^{(d-3)}(z)$, so we get anther two starting points for iteration of $N_p(z)$ to find the two roots which are $x_{\frac{d-1}{2}}, x_{\frac{d+3}{2}}$. Now we follow the same steps until we find the zeros of $p''(z)$ then we have two starting points for iteration of N_p to find the two roots x_2 , x_{d-1}. Finally for the two roots x_1 , x_d we can find them as in case one.

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