ON THE EXTREME POINTS OF CLASSES OF UNIVALENT FUNCTIONS

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Abstract :
The primary motivation of the paper is to define a new class $\emptyset_q^m(\gamma_1,\gamma_2,\alpha,d_n)$ which consists of univalent functions and study some properties of certain subclass of analytic functions which defined by generalized derivative operator on the open unit disc .

الملخص (باللغة العربية):

الدافع الأساسي لهذا البحث هو تعريف فئة جديدة $\emptyset_q^m(\gamma_1,\gamma_2,\alpha,d_n)$ والتي تتكون من دوال أحادية التكافؤ ودراسة بعض خصائص فئة معينة من الدوال التحليلية التي تم تعريفها بواسطة عامل مشتق معمم على قرص الوحدة المفتوح .

Keywords: Analytic function**,** Hadamard product, Unit disk, Convex's order, Convex linear combination.

1. Introduction

Let the functions f(z) in the open unit disk $\mathbb{U}=\{z:|z|<1; z\in\mathbb{C}\}$ belong to the class $\mathcal{A}$ which is taking of the form :

$$f(z)=z+\sum_{k=2}^{\infty}a_k z^k \quad (z\in\mathbb{U}) \ , \quad (1)$$

where a_k is a complex number .

The Hadamard product of two analytic functions f and g denoted by $f*g$, where $f(z)$ *form* (1) *and* $g(z)=z+\sum_{k=2}^{\infty}b_k z^k$; $(z\in\mathbb{U})$,is defined by

$$(f*g)(z)=f(z)*g(z)=z+\sum_{k=2}^{\infty}a_k b_k z^k \quad , \quad (z\in\mathbb{U}).$$

And by using this product, Amer and Darus (Amer & Darus, 2011) they have recently introduced a new generalized derivative operator**.**

Definition 1:

For $f \in \mathcal{A}$ the operator $I^m(\lambda_1, \lambda_{2,}\ell, n)$ is defined by $I^m(\lambda_1, \lambda_{2,}\ell, n): \mathcal{A} \longrightarrow \mathcal{A}$.

$$I^m(\lambda_1, \lambda_{2,}\ell, n)f(z) = \varphi^m(\lambda_1, \lambda_2, \ell)(z) * R^n f(z) \quad (z \in \mathbb{U}) \ ,$$

where $m \in N_0 = \{0,1,2, \dots\dots\}$ and $\lambda_2 \geq \lambda_1 \geq 0,\ \ell \geq 0.$ and $R^n f(z)$ denotes the Ruseheweyh derivative operator and given by

$$R^n f(z) = z + \sum_{k=2}^{\infty} c(n,k)\ a_k z^k \ , \ (n \in N_0, z \in \mathbb{U})$$

where $c(n,k) = \frac{(n+1)_{k-1}}{(1)_{k-1}}$.

If $f(z)$ given by (1), then we easily find from

$$I^m(\lambda_1, \lambda_{2,}\ell, n)f(z) = z + \sum_{k=2}^{\infty} \frac{(1+\lambda_1(k-1)+\ell)^{m-1}}{(1+\ell)^{m-1}\left(1+\lambda_2(k-1)\right)^m} c(n,k) a_k z^k, \longrightarrow (2)$$

where $n, m \in N_0 = \{0,1,2, \dots\dots\}\ and\ \lambda_2 \geq \lambda_1 \geq 0,\ \ell \geq 0.$

Some special cases of this operator includes :

- The Ruscheweyh derivative operator (Ruscheweyh, 1975) in the cases :
$$I^1(\lambda_1, 0, l, n) \equiv I^1(\lambda_1, 0,0, n) \equiv I^1(0,0, l, n) \equiv I^0(0, \lambda_2, 0, n) \equiv I^0(0,0,0, n) \equiv I^{m+1}(0,0, l, n) \equiv I^{m+1}(0,0,0, n) \equiv \mathbb{R}^n.$$
- The Salagean derivative operator(Salagean, 2006):
$$I^{m+1}(1,0,0,0) \equiv D^n.$$
- The generalized Salagean derivative operator introduced by Al-Oboudi (Al-Oboudi, 2004) :
$$I^{m+1}(\lambda_1, 0,0,0) \equiv D_\beta^n.$$

Using simple computation one obtains the next result

$$(\ell+1)I^{m+1}(\lambda_1, \lambda_{2,}\ell, n)f(z) = (1+\ell-\lambda_1)(I^m(\lambda_1, \lambda_{2,}\ell, n) * \varphi^1(\lambda_1, \lambda_2, \ell)(z))f(z) + \lambda_1 z\left(I^m\left(\lambda_1, \lambda_{2,}\ell, n\right) * \varphi^1(\lambda_1, \lambda_2, \ell)f(z)\right)', \longrightarrow (3)$$

where $(z \in \mathbb{U})$ and $\varphi^1(\lambda_1, \lambda_2, \ell)(z)$ analytic function given by

$$\varphi^1(\lambda_1,\lambda_2,\ell)(z)=z+\sum_{k=2}^{\infty}\frac{1}{\left(1+\lambda_2(k-1)\right)}\ z^k.\longrightarrow (4)$$

Now, from equation (2) and (4) ,we have

$$\left(I^m(\lambda_1,\lambda_{2,}\ell,n)*\varphi^1(\lambda_1,\lambda_2,\ell)f(z)\right)'$$
$$=\left(z+\sum_{k=2}^{\infty}\frac{(1+\lambda_1(k-1)+\ell)^{m-1}}{(1+\ell)^{m-1}\left(1+\lambda_2(k-1)\right)^m}c(n,k)a_kz^k*z+\sum_{k=2}^{\infty}\frac{1}{\left(1+\lambda_2(k-1)\right)}\ z^k\right)'$$
$$=\left(z+\sum_{k=2}^{\infty}\frac{(1+\lambda_1(k-1)+\ell)^{m-1}}{(1+\ell)^{m-1}\left(1+\lambda_2(k-1)\right)^m}c(n,k)a_kz^k\right)'$$
$$=\left(I^m(\lambda_1,\lambda_{2,}\ell,n)f(z)\right)'$$

So, from equation (3), we obtain

$$z\left(I^m(\lambda_1,\lambda_{2,}\ell,n)f(z)\right)'=\frac{(\ell+1)}{\lambda_1}I^{m+1}(\lambda_1,\lambda_{2,}\ell,n)f(z)-\frac{(1+\ell-\lambda_1)}{\lambda_1}(I^m(\lambda_1,\lambda_{2,}\ell,n)f(z).\longrightarrow (5)$$

Definition 2:

Let $\lambda_2\geq\lambda_1\geq 0$, $l\geq 0$,$0\leq\gamma<1$, and $f\in$ T , such that $I^m(\lambda_1,\lambda_2,l,n)f(z)$ for z ∈ $\mathbb{U}$. We say that $f\in\emptyset_q^m(\lambda_1,\lambda_2,l,n,\gamma)$ if and only if

$$\emptyset_q^m(\lambda_1,\lambda_2,l,n,\gamma)=\left\{f\in\mathcal{A}:Re\left\{\frac{z\nabla_q(I^m(\lambda_1,\lambda_2,l,n)f(z))}{I^m(\lambda_1,\lambda_2,l,n)f(z)}\right\}>\gamma\ ,\right\}$$

When ∇_q denote to the Jackson's q- derivative [2] .

Now, we define the class given by $\emptyset_q^m(\lambda_1,\lambda_2,l,n,\gamma)$.

The functions that belong to this class satisfy the following properties:

1. The function $f\in\emptyset_q^m(\lambda_1,\lambda_2,l,n,\gamma)$ if and only if

$$\sum_{k=2}^{\infty}([k]_q-\gamma)\Psi_{q,k}^m(\lambda_1,\lambda_2)a_k\leq 1-\gamma. \qquad (6)$$

When

$$a_i = \frac{d_i(1-\gamma)}{\big([i]_q-\gamma\big)\Psi_{q,k}^m(\lambda_1,\lambda_2,l,n)}\ , \quad \text{for } i = 2,3,\dots,n\ . \tag{7}$$

And $[k]_q = \frac{1-q^k}{1-q}$, $[0]_q = 0.$

2. If $f \in \emptyset_q^m(\lambda_1,\lambda_2,l,n,\gamma)$, such that

$$f(z) = z - \frac{1-\gamma}{\big([k]_q-\gamma\big)\Psi_{q,k}^m(\lambda_1,\lambda_2,l,n)} z^k \quad . \tag{8}$$

Then we have

$$a_k \le \frac{1-\gamma}{\big([k]_q-\gamma\big)\Psi_{q,k}^m(\lambda_1,\lambda_2,l,n)} \quad . \tag{9}$$

Definition 3:

Let $\emptyset_q^m(\lambda_1,\lambda_2,l,n,\gamma,d_n)$ be the subclass of $\emptyset_q^m(\lambda_1,\lambda_2,l,n,\gamma)$ consisting of functions of the form

$$f(z) = z - \sum_{i=2}^{n} \frac{d_i(1-\gamma)}{([i]_q-\gamma)\Psi_{q,k}^m(\lambda_1,\lambda_2,l,n)} z^i - \sum_{k=n+1}^{\infty} a_k z_k . \tag{10}$$

Where $0 \le d_i \le 1$ and $\sum_{i=2}^{n} d_i \le 1$.

This subclass satisfied the following:

1. Let $f(z) \in \emptyset_q^m(\lambda_1,\lambda_2,l,n,\gamma)$, Then $f(z) \in \emptyset_q^m(\lambda_1,\lambda_2,l,n,\gamma,d_n)$ if and only if

$$\sum_{k=n+1}^{\infty} ([k]_q-\gamma)\Psi_{q,k}^m(\lambda_1,\lambda_2,l,n)a_k \le (1-\gamma)(1-\sum_{i=2}^{n} d_i). \tag{11}$$

2. If $f(z) \in \emptyset_q^m(\lambda_1,\lambda_2,l,n,\gamma,d_n)$, and satisfied equations (7) and (10), then

$$a_{\mathrm{k}} \le \frac{(1-\gamma)(1-\sum_{i=2}^{n} d_i)}{\big([k]_q-\gamma\big)\Psi_{q,k}^m(\lambda_1,\lambda_2,l,n)}, \qquad k \ge n+1 \quad . \tag{12}$$

3. *If $f(z) \in \emptyset_q^m(\lambda_1,\lambda_2,l,n,\gamma,d_n)$, then*

$$\sum_{\mathrm{k=n+1}}^{\infty}[k]_q a_{\mathrm{k}} \leq \frac{[n+1]_q(1-\gamma)(1-\sum_{i=2}^{n} d_i)}{([n+1]_q-\gamma)\Psi_{q,k}^{m}(\lambda_1,\lambda_2,l,n+1)}. \qquad (13)$$

$$f(z) = z - \sum_{i=2}^{n}\frac{d_i(1-\gamma)}{([i]_q-\gamma)\Psi_{q,k}^{m}(\lambda_1,\lambda_2,l,n)}z^i - \sum_{k=n+1}^{\infty}\frac{(1-\gamma)\sum_{i=2}^{n} d_i}{([k]_q-\gamma)\Psi_{q,k}^{m}(\lambda_1,\lambda_2,l,n)}z^i. \quad (14)$$

All the properties mentioned previously have been proven in a previous study [2].

Main Result:

Theorem *1*

If t_0 is the largest value for which

$$\sum_{i=2}^{n}\frac{[i]_q d_i(1-\alpha)\big([i-1]_q+1-\tau\big)}{\big([i]_q-\alpha\big)\Psi_{q,i}^{m}(\gamma_1,\gamma_2)}{t_0}^{i-1} + \frac{[k]_q\big([k-1]_q+1-\tau\big)(1-\alpha)(1-\sum_{i=2}^{n} d_i)}{\big([k]_q-\alpha\big)\Psi_{q,k}^{m}(\gamma_1,\gamma_2)}{t_0}^{k-1} \leq 1-\tau \qquad (15)$$

And $f(z) \in \emptyset_q^m(\gamma_1,\gamma_2,\alpha,d_n)$ be defined by (9), and satisfied (14), then $f(z)$ is covex of order $\tau, (0 \leq \tau < 1)\ , in\ (0 < |z| < t_0)$, for $k \geq n+1$.

Proof:

To prove that , the order of convex to $f(z) \in \emptyset_q^m(\gamma_1,\gamma_2,\alpha,d_n)$ is τ, we have to show that

$$\left|\frac{zf''(z)}{f'(z)}\right| \leq 1-\tau\,, \qquad (|z| < t_0). \qquad (16)$$

i.e

$$\left|\frac{zf''(z)}{f'(z)}\right| \le \frac{\sum_{i=2}^{n}\frac{[i]_q d_i(1-\alpha)[i-1]_q}{([i]_q-\alpha)\Psi_{q,i}^m(\gamma_1,\gamma_2)}|z|^{i-1}+\sum_{k=n+1}^{\infty}[k]_q[k-1]_q a_k|z|^{k-1}}{1-\sum_{i=2}^{n}\frac{[i]_q d_i(1-\alpha)}{([i]_q-\alpha)\Psi_{q,i}^m(\gamma_1,\gamma_2)}|z|^{i-1}-\sum_{k=n+1}^{\infty}[k]_q a_k|z|^{k-1}}. \qquad (17)$$

Now, from (9) , (15), when $(|z|<t_0)$ *, we have*

$$\sum_{i=2}^{n}\frac{[i]_q d_i(1-\alpha)([i-1]_q+1-\tau)}{([i]_q-\alpha)\Psi_{q,i}^m(\gamma_1,\gamma_2)}|z|^{i-1} + \sum_{k=n+1}^{\infty}[k]_q([k-1]_q+1-\tau)a_k|z|^{k-1} \le 1-\tau. \qquad (18)$$

Hence by (8) and (18) we have

$$\sum_{i=2}^{n}\frac{[i]_q d_i(1-\alpha)([i-1]_q+1-\tau)}{([i]_q-\alpha)\Psi_{q,i}^m(\gamma_1,\gamma_2)}|z|^{i-1} + \frac{[k]_q\left([k-1]_q+1-\tau\right)(1-\sum_{i=2}^{n}d_i)}{([k]_q-\alpha)\Psi_{q,k}^m(\gamma_1,\gamma_2)}|z|^{k-1} \le 1-\tau. \qquad (19)$$

Theorem 2 :

The class $\emptyset_q^m(\gamma_1,\gamma_2,\alpha,d_n)$ is closed under convex linear combination.

Proof:

Let $f(z)$ be defined by (10). Define the function $g(z)$ by

$$g(z)=z-\sum_{i=2}^{n}\frac{d_i(1-\alpha)}{([i]_q-\alpha)\Psi_{q,i}^m(\gamma_1,\gamma_2)}z^i+\sum_{k=n+1}^{\infty}b_k z^k. \qquad (20)$$

Suppose that $f(z)$, $(z) \in \emptyset_q^m(\gamma_1,\gamma_2,\alpha,d_n)$, let

$$H(z)=\beta f(z)+(1-\beta)g(z) \qquad (0\le\beta\le 1), \qquad (21)$$

We have to prove that $H(z) \in \emptyset_q^m(\gamma_1,\gamma_2,\alpha,d_n)$, Since

$$H(z) = z - \sum_{i=2}^{n} \frac{d_i(1-\alpha)}{([i]_q - \alpha)\Psi_{q,i}^{m}(\gamma_1,\gamma_2)} z^i$$

$$+ \sum_{k=n+1}^{\infty} (\beta a_k + (1-\beta) b_k) z^k, \qquad (22)$$

Then

$$\sum_{k=n+1}^{\infty} \frac{\big([k]_q - \alpha\big)\Psi_{q,k}^{m}(\gamma_1,\gamma_2)}{(1-\alpha)} (\beta a_k + (1-\beta) b_k) \le \left(1 - \sum_{i=2}^{n} d_i\right), \qquad (23)$$

From (8) , we conclude that $H(z) \epsilon \ \emptyset_q^m(\gamma_1,\gamma_2,\alpha,d_n)$,

So, the class $\emptyset_q^m(\gamma_1,\gamma_2.\alpha,d_n)$ is closed under convex linear combination.

Corollary 1:

Let $f_j(z) \in \emptyset_q^m(\gamma_1,\gamma_2,\alpha,d_n)$, such that

$$f_j(z) = z - \sum_{i=2}^{n} \frac{d_i(1-\alpha)}{\big([i]_q - \alpha\big)\Psi_{q,i}^{m}(\gamma_1,\gamma_2)} z^i - \sum_{k=n+1}^{\infty} a_{kj} z^k. \quad j = 1,2,\dots,s \ . \qquad (19)$$

Then the function $B(z)$ defined by

$$B(z) = \sum_{j=1}^{s} e_j f_j(z) \qquad \big(e_j \ge 0\big), \qquad (25)$$

is also in the class $\emptyset_q^m(\gamma_1,\gamma_2,\alpha,d_n)$, where $\sum_{j=1}^{s} e_j = 1$.

Corollary. 2

Let the function $f_j(z)$ defined by (24) be in the class $\emptyset_q^m(\gamma_1,\gamma_2,\alpha,d_n)$ for each $j = 1,2,\dots,s$, then the function $C(z)$ defined by

$$C(z) = z - \sum_{i=2}^{n} \frac{d_i(1-\alpha)}{([i]_q - \alpha)\Psi_{q,i}^{m}(\gamma_1,\gamma_2)} z^i - \sum_{k=n+1}^{\infty} b_k z^k. \qquad (26)$$

Also be in the class $\emptyset_q^m(\gamma_1,\gamma_2,\alpha,d_n)$, where

$$b_k = \frac{1}{s}\sum_{i=2}^{s} a_{ki} \,.$$

The corollaries (1), (2) are generalize theorem (2).

Theorem 3

Let

$$f_n(z) = z - \sum_{i=2}^{n} \frac{d_i(1-\alpha)}{([i]_q - \alpha)\Psi_{q,i}^{m}(\gamma_1,\gamma_2)} z^i \,, \qquad (27)$$

And

$$f_k(z) = z - \sum_{i=2}^{n} \frac{d_i(1-\alpha)}{([i]_q - \alpha)\Psi_{q,i}^{m}(\gamma_1,\gamma_2)} z^i - \sum_{k=n+1}^{\infty} \frac{(1-\alpha)(1-\sum_{i=2}^{n} d_i)}{([k]_q - \alpha)\Psi_{q,k}^{m}(\gamma_1,\gamma_2)} z^k \quad (28)$$

For $k > n+1$. Then the function $f(z)$ is in the class $\emptyset_q^m(\gamma_1,\gamma_2.\propto, d_n)$ if and only if it can be expressed in the form

$$f(z) = \sum_{j=n}^{\infty} p_j f_j(z) \quad where \;\; \left(p_j \geq 0\,,\;\; j \geq n\right), \;\; and \;\; \sum_{j=n}^{\infty} p_j = 1. \qquad (29)$$

Proof:

Let the function $f(z)$ can be expressed in the form (30). Then we get

$$f(z) = z - \sum_{i=2}^{n} \frac{d_i(1-\alpha)}{\left([i]_q - \alpha\right)\Psi_{q,i}^{m}(\gamma_1,\gamma_2)} z^i - \sum_{k=n+1}^{\infty} \frac{p_k(1-\alpha)(1-\sum_{i=2}^{n} d_i)}{\left([k]_q - \alpha\right)\Psi_{q,k}^{m}(\gamma_1,\gamma_2)} z^k \quad (30)$$

Since

$$\sum_{k=n+1}^{\infty} \frac{p_k(1-\alpha)(1-\sum_{i=2}^{n} d_i)([k]_q - \alpha)\Psi_{q,k}^{m}(\gamma_1,\gamma_2)}{\left([k]_q - \alpha\right)\Psi_{q,k}^{m}(\gamma_1,\gamma_2)(1-\alpha)}$$

$$= \left(1 - \sum_{i=2}^{n} d_i\right) \sum_{k=n+1}^{\infty} p_k$$

$$= \left(1 - \sum_{i=2}^{n} d_i\right)(1 - p_n) \leq \left(1 - \sum_{i=2}^{n} d_i\right) \qquad (31)$$

Then $f(z)\epsilon \;\; \emptyset_q^m(\gamma_1,\gamma_2,\alpha,d_n)$.

Conversely, assuming that $f(z)$ defined by (10), be in class $\emptyset_q^m(\gamma_1,\gamma_2,\alpha,d_n)$ which satisfies (12) for $k \geq n+1,$ we obtain

$$p_k = \frac{\left([k]_q - \alpha\right)\Psi_{q,k}^m(\gamma_1,\gamma_2)}{(1-\alpha)\left(1-\sum_{i=2}^{n} d_i\right)}\ a_k \leq 1\ ,$$

And

$$p_k = 1 - \sum_{k=n+1}^{\infty} d_i$$

Corollary. 3

The extreme point of the class $\emptyset_q^m(\gamma_1,\gamma_2,\alpha,d_n)$ are the function $f_k(z)$ $(k \geq n)$ given by in theorem 3 .

Conclusion

in this work , by generalized derivative operator we define the class given by $\emptyset_q^m(\lambda_1,\lambda_2,l,n,\gamma)$ of analytic functions, and properties are derived .

Many other work on analytic functions related to derivative operator can be read in (Shmella & Amer, 2024a), (Amer & Alabbar, 2017) ,(Amer, Alshbear, & Alabbar, 2017) ,(Shmella & Amer, 2024b). There are times, functions are associated with create new classes and linear operators). Many results are considered with numerous properties are solved and obtained.

References

[1] Abufares ,F .and Amer .A , (2023) , Some Applications of Fractional Differential Operators in the Field of Geometric Function .Theory,Conference on basic sciences and their applications .

[2] Hanan Mohamed Almarwm, Aisha Ahmed Amer and Samah Khairi Ajaib , (2024), The Coefficient Estimates for A New Class of Analytic Functions with Negative Coefficients. المجلد التاسع العدد الثاني , مجلة جامعة بني وليد للعلوم الإنسانية والتطبيقية

[2] Al-Oboudi, F.M. (**2004**) , On univalent functions defined by a generalized Sălăgean operator. International Journal of Mathematics and Mathematical Sciences,. p. 1429-1436.

[3] Al-Shaqsi, K. and Darus . M , (2008), Differential subordination with generalized derivative operator. Int. J. Comp. Math. Sci,**2** p. 75-78.
[4] Al-Abbadi, M . and Darus . M, (2009) , Differential subordination for new generalised derivative operator. Acta Universitatis Apulensis. Mathematics-Informatics, **20**. p. 265-280.
[5] Alabbar. N, Darus. M & Amer.A. (2023), Coefficient Inequality and Coefficient Bounds for a New Subclass of Bazilevic Functions. Journal of Humanitarian and Applied Sciences, **8** -496-506.
[6]Amer, A.A. and Darus. M , (2011), On some properties for new generalized derivative operator. Jordan Journal of Mathematics and Statistics (JJMS), **4** -(2): p. 91-101.
[7] Amer.A , Darus , M and Alabbar .N .(2024) , Properties For Generalized Starlike and Convex Functions of Order , Fezzan University scientific Journal.
[8] Annby. M. H and Mansour. Z. S, **(**2012) , q−Fractional Calculus Equations. Lecture Notes in Mathematics., Vol. **2056,** Springer, Berlin.
[9] Aouf. M. K. , Darwish. H. E and S˘al˘agean . G. S, (2001), , On a generalization of starlike functions with negative coefficients, Math., Tome **66** - no. 1, 3–10.
[10] Catas, A. and Borsa .E , (2009). On a certain differential sandwich theorem associated with a new generalized derivative operator. General Mathematics, **17**-p. 83-95.
[11] Frasin . B.A (2006), Family of analytic functions of complex order, Acta Math. Acad. Paedagogicae Ny , **22** -no. 2, 179–191.
[12] Ismail. M.E, Merkes. M.E , and David. M.E, (1990), A generalization of starlike functions, Transit. Complex Variables, Theory Appl. **61** ,77–84.
[13] Jackson. F. H, (1908), On q−functions and a certain difference operator, Trans. R. Soc. Edinb., **46**-253–281.
[14] Ruscheweyh, S. **(** 1975). New criteria for univalent functions. Proceedings of the American Mathematical Society, **49** -(1): p. 109-115.
[15] Salagean, G.S. (1981), Subclasses of univalent functions, Complex analysis-fifth Romanian-Finnish seminar, **1** , 362-372. Lecture Notes in Math, 1983.
[16] SALEH .Z. M. , MOSTAFA. A. O. (2022), Class of analytic univalent functions with fixed finite negative coefficients defined by q-analogue difference operator, Jordan Journal of Mathematics and Statistics (JJMS), 15(4A), 2022, pp 955- 965, DOI: https://doi.org/10.47013/15.4.11

[17] Shaqsi, K. and Darus . M , **(**2008) ,An operator defined by convolution involving the polylogarithms functions. Journal of Mathematics and Statistics, **4**. (1): p. 46.
[18] Shmella. E.K and Amer. A.A , (2023) , Estimation of the Bounds of Univalent Functional of Coefficients Apply the Subordination Method, The Academic Open Journal Of Applied And Human Sciences ,**5**,(2709-3344), issue (1)
[19] Silverman. H , (1975), Univalent functions with negative coefficients, Proc. Amer. Math. Soc. 51 ,no. 1, 109–116.