A Note on Triple Natural transform and its application to some kind of Volterra integro-differential equations

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Abstract
In this article, we discuss the application of different properties of the triple natural transform to solve the linear Volterra integro-differential equations in three dimensions.

Keyword: Triple Natural transform, Inverse triple Natural transform, Partial integral-differential equations

1. Introduction

Linear integral equations are used to model many problems in engineering, chemistry, physics and many other disciplines of study. It is well known that most integro-differential equations give solutions in a closed form. It is therefore important to propose new methods of finding solutions to various integro- differential equations [1–2].

Many various methods such as double Laplace transform [3], double Sumudu transform [4], double Natural transform [5], introduced by many researchers to find the solution of partial differential equations. In addition to these, Triple Natural transform was defined, related theorems and properties were presented in [6].

One example of linear integro-differential equations is the three-dimension linear Volterra integro-differential equations (LVIDEs), which are obtained in the course of modeling engineering applications. The general three-dimensional LVIDEs are given in the following form (see [7,8])

$$\frac{\partial^3 f(x,y,t)}{\partial x \partial y \partial t} + f(x,y,t) = g(x,y,t) + \int_{x_0}^{x}\int_{y_0}^{y}\int_{t_0}^{t} H\big(x,y,t,k,r,s,f(k,r,s)\big)dkdrds, \qquad (1.1)$$

where, $f(x,y,t)$ are the unknown function, and the function H and g are analytic in the domain of interest.

The triple natural transform defined of the function $f(x,y,t)$ as

$$N_+^3\{f(x,y,t)\} = R_+^3[(\alpha,\beta,\gamma),(u,v,w)]$$
$$= \int_0^{\infty}\int_0^{\infty}\int_0^{\infty} e^{-(\alpha x+\beta y+\gamma t)} f(ux,vy,wt)dxdydt. \qquad (1.2)$$

The equation (1.2) can be written as

$$N_{+}^{3}\{f(x,y,t)\} = \frac{1}{uvw}\int_{0}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty} e^{-\left(\frac{\alpha x}{u}+\frac{\beta y}{v}+\frac{\gamma t}{w}\right)} f(x,y,t)\, dxdydt. \qquad (1.3)$$

The inverse of triple natural transform is given as the formula

$$N_{+}^{-3}[N_{+}^{3}\{f(x,y,t)\}] = f(x,y,t)$$

$$= \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{\frac{\alpha x}{u}} d\alpha \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} e^{\frac{\beta y}{v}} d\beta \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\frac{\gamma t}{w}} N_{+}^{3}\{f(x,y,t)\} d\gamma. \qquad (1.4)$$

The structure of this paper is organized as follows: In section 2, the existence and uniqueness of the solution of the equation (1) are given. Basic properties of triple natural transform are studied in section 3. In section 4, three examples to solve linear volterra integro-differential equations in three-dimensions are applied. Finally the conclusion of this work is obtained.

2. On the existence of the solution linear Volterra integro-differential equations of the three-dimensional

In this section, we recall the conditions necessary for the existence and uniqueness of the solution of the equation (1.1), on the complete metric space of complex valued continuous functions as follows (see [8])

$$M = [C(S,d)], d(g,z) = sup\{|h(x,y,t) - z(x,y,t) : (x,y,t) \in S|\}$$

where $S = [0, 1] \times [0, 1] \times [0, 1]$.

Theorem 2.1. Let g and H be continuous functions on $[0, 1]^3$ and $[0, 1] \times [0, 1] \times \mathbb{C}$ respectively and there exists a nonnegative constant $L \leq 1$ such that

$$\left|H\big(x,y,t,k,r,s,f(k,r,s)\big) - H\big(x,y,t,k,r,s,h(k,r,s)\big)\right| \leq L|f(k,r,s) - h(k,r,s)|$$

Then

$$f(x,y,t) = g(x,y,t) + \int_{0}^{x}\int_{0}^{y}\int_{0}^{t} H\big(x,y,t,k,r,s,f(k,r,s)\big) dkdrds,$$

has only one continuous solution f on S.

Proof. For the proof see [8].

Corollary 2.2. If the hypothesis of the Theorem 2.1 holds, then the equation (1.1)

$$\frac{\partial^3 f(x,y,t)}{\partial x \partial y \partial t} + f(x,y,t) = g(x,y,t) + \int_{x_0}^{x}\int_{y_0}^{y}\int_{t_0}^{t} H\big(x,y,t,k,r,s,f(k,r,s)\big) dkdrds,$$

with initial condition

$$f(0,0,0)=h_0, f(x,0,0)=h_1(x), f(0,y,0)=h_2(y), f(0,0,t)=h_3(t),$$
$$f(x,y,0)=h_4(x,y), f(x,0,t)=h_5(x,t), f(0,y,t)=h_6(y,t)$$

So by Theorem 2.1 the equation has a unique continuous solution. (see [9]).

3. Properties of the triple natural transform

Some properties of the triple natural transform are stated in this section which will be used in the following section to solve the LIVIDEs.

Linearity of the triple natural transform:

For any constants a, b, c such that

$$N_+^3\left[a\, f(x,y,t)+b\, g(x,y,t)+c\, h(x,y,t)\right] = aN_+^3\left[f(x,y,t)\right]+b\, N_+^3\left[g(x,y,t)\right]+cN_+^3\left[h(x,y,t)\right]$$

Operational formulas:

1. $N_+^3\left[\frac{\partial^3}{\partial x\partial y\partial t} f(x,y,t)\right]=\frac{\alpha\beta\gamma}{uvw} N_+^3\left[f(x,y,t)\right]-\frac{\alpha\beta}{uv} N_+^3\left[f(x,y,0)\right]-\frac{\alpha\gamma}{uw} N_+^3\left[f(x,0,t)\right]-\frac{\beta\gamma}{vw} N_+^3\left[f(0,y,t)\right]+\frac{\alpha}{u} N_+^3\left[f(x,0,0)\right]+\frac{\beta}{v} N_+^3\left[f(0,y,0)\right]+\frac{\gamma}{w} N_+^3\left[f(0,0,t)\right]-\frac{N_+^3\left[f(0,0,0)\right]}{uvw}$

2. $N_+^3\left[\int_0^x\int_0^y\int_0^t f(x,y,t)dxdydt\right]=\frac{uvw}{\alpha\beta\gamma} N_+^3\left[f(x,y,t)\right]$

4. Application to Volterra integro-differential equations

In this section, we apply the various proerties of triple Natural transform to determine the solution of linear Volterra integro-differential equations. We consider the two following examples and work out their respective solutions.

Example 1

Consider the linear volterra integro-differential equation

$$\frac{\partial^3 f(x,y,t)}{\partial x\partial y\partial t}+f(x,y,t)=xyt+1-\frac{x^2\, y^2 t^2}{8}+\int_0^x\int_0^y\int_0^t f(k,r,s)dkdrds, \quad (4.1)$$

With initial condition $\quad f(x,y,0)=f(x,0,t)=f(0,y,t)=0. \quad (4.2)$

Solution.

By taking the triple Natural transform on both sides of (4.1), and double Natural transform of initial conditions equation (4.2), then we obtain

$$N_+^3\left[\frac{\partial^3 f(x,y,t)}{\partial x\partial y\partial t}\right]+N_+^3\left[f(x,y,t)\right]=N_+^3\left[xyt+1-\frac{x^2\, y^2 t^2}{8}+\int_0^x\int_0^y\int_0^t f(k,r,s)dkdrds\right]$$

$\frac{\alpha\beta\gamma}{uvw}\ N_+^3\,[f(x,y,t)] - \frac{\alpha\beta}{uv}\ N_+^3\,[f(x,y,0)] - \frac{\alpha\gamma}{uw}\ N_+^3\,[f(x,0,t)] - \frac{\beta\gamma}{vw}\ N_+^3\,[f(0,y,t)] +$
$\frac{\alpha}{u}\ N_+^3\,[f(x,0,0)] + \frac{\beta}{v}\ N_+^3\,[f(0,y,0)] + \frac{\gamma}{w}\ N_+^3\,[f(0,0,t)] - \frac{N_+^3\,[f(0,0,0)]}{uvw} + N_+^3\,[f(x,y,t)] =$
$\frac{uvw}{\alpha^2\beta^2\gamma^2} + \frac{1}{\alpha\beta\gamma} - \frac{u^2v^2w^2}{\alpha^3\beta^3\gamma^3} + \frac{uvw}{\alpha\beta\gamma}\ N_+^3\,[f(x,y,t)],$ (4.3)

and

$$N_+^2\,[f(x,y,0)] = N_+^2\,[f(x,0,t)] = N_+^2\,[f(0,y,t)] = 0,\ \ N_+^3\,[f(0,0,0)] = 0\,. \qquad (4.4)$$

Substituting (4.4) in (4.3), and using the fact

$N_+^3\,[f(x,y,0)] = N_+^2\,[f(x,y,0)],\qquad N_+^3\,[f(x,0,t)] = N_+^2\,[f(x,0,t)],$

$N_+^3\,[f(0,y,t)] = N_+^2\,[f(0,y,t)]$, then we get

$$\left(\frac{\alpha\beta\gamma}{uvw} + 1 - \frac{uvw}{\alpha\beta\gamma}\right) N_+^3\,[f(x,y,t)] = \frac{uvw}{\alpha^2\beta^2\gamma^2}\left(1 + \frac{\alpha\beta\gamma}{uvw} - \frac{uvw}{\alpha\beta\gamma}\right)$$

$$N_+^3\,[f(x,y,t)] = \frac{uvw}{\alpha^2\beta^2\gamma^2}$$

Appling triple Natural transform of both sides, then we have

$$f(x,y,t) = xyt$$

Example 2

Consider the linear volterra integro-differential equation

$$\frac{\partial^3 f(x,y,t)}{\partial x\partial y\partial t} + f(x,y,t) = \frac{x^2yt + x\,y^2t + xyt^2}{2} + x + y + t - \int_0^x\int_0^y\int_0^t f(k,r,s)dkdrds, \quad (4.5)$$

with initial condition

$f(x,y,0) = x + y,\ \ f(x,0,t) = x + t,\ \ f(0,y,t) = y + t,\ \ f(x,0,0) = x,\ \ f(0,y,0) = y,$
$f(0,0,t) = t\ ,\ \ f(0,0,0) = 0$ (4.6)

Solution.

We apply the triple Natural transform on both sides of (4.5), and double Natural transform of initial conditions equation (4.6), then we have

$$N_+^3\left[\frac{\partial^3 f(x,y,t)}{\partial x\partial y\partial t}\right] + N_+^3\,[f(x,y,t)]$$
$$= N_+^3\left[\frac{x^2yt + x\,y^2t + xyt^2}{2} + x + y + t - \int_0^x\int_0^y\int_0^t f(k,r,s)dkdrds\right]$$

$$\frac{\alpha\beta\gamma}{uvw}\,N_+^3\,[f(x,y,t)]-\frac{\alpha\beta}{uv}\,N_+^3\,[f(x,y,0)]-\frac{\alpha\gamma}{uw}\,N_+^3\,[f(x,0,t)]-\frac{\beta\gamma}{vw}\,N_+^3\,[f(0,y,t)]$$
$$+\frac{\alpha}{u}\,N_+^3\,[f(x,0,0)]+\frac{\beta}{v}\,N_+^3\,[f(0,y,0)]+\frac{\gamma}{w}\,N_+^3\,[f(0,0,t)]-\frac{N_+^3\,[f(0,0,0)]}{uvw}$$
$$+N_+^3\,[f(x,y,t)]$$
$$=\frac{u^3v^2w^2}{\alpha^3\beta^2\gamma^2}+\frac{u^2v^3w^2}{\alpha^2\beta^3\gamma^2}+\frac{u^2v^2w^3}{\alpha^2\beta^2\gamma^3}+\frac{u^2vw}{\alpha^2\beta\gamma}+\frac{\mathrm{u}v^2\mathrm{w}}{\alpha\beta^2\gamma}+\frac{uvw^2}{\alpha\beta\gamma^2}$$
$$-\frac{uvw}{\alpha\beta\gamma}\,N_+^3\,[f(x,y,t)], \qquad (4.7)$$

and

$$N_+^3\,[f(x,y,0)]=\frac{u^2\mathrm{v}}{\alpha^2\beta}+\frac{u\,v^2}{\alpha\beta^2}\,,\ N_+^3\,[f(x,0,t)]=\frac{u^2w}{\alpha^2\gamma}+\frac{u\,w^2}{\alpha\gamma^2}\,,\ N_+^3\,[f(0,y,t)]\ =\ \frac{v^2w}{\beta^2\gamma}+\frac{v\,w^2}{\beta\gamma^2}$$

$$N_+^3\,[f(x,0,0)]=\frac{u^2}{\alpha^2},\ N_+^3\,[f(0,y,0)]=\frac{v^2}{\beta^2}\,,\ N_+^3\,[f(0,0,t)]=\frac{w^2}{\gamma^2},\ N_+^3\,[f(0,0,0)]=0\,. \qquad (4.8)$$

Substituting the equation (4.8) in the equation (4.7), then we obtain

$$\left(\frac{\alpha\beta\gamma}{uvw}+1+\frac{uvw}{\alpha\beta\gamma}\right)N_+^3\,[f(x,y,t)]$$
$$=\frac{u}{\alpha}+\frac{v}{\beta}+\frac{w}{\gamma}+\frac{u^3v^2w^2}{\alpha^3\beta^2\gamma^2}+\frac{u^2v^3w^2}{\alpha^2\beta^3\gamma^2}+\frac{u^2v^2w^3}{\alpha^2\beta^2\gamma^3}+\frac{u^2vw}{\alpha^2\beta\gamma}+\frac{\mathrm{u}v^2\mathrm{w}}{\alpha\beta^2\gamma}+\frac{uvw^2}{\alpha\beta\gamma^2}$$
$$=\left(\frac{\alpha\beta\gamma}{uvw}+\frac{uvw}{\alpha\beta\gamma}+1\right)\left(\frac{u^2vw}{\alpha^2\beta\gamma}+\frac{\mathrm{u}v^2\mathrm{w}}{\alpha\beta^2\gamma}+\frac{uvw^2}{\alpha\beta\gamma^2}\right).$$

then

$$N_+^3\,[f(x,y,t)]=\frac{u^2vw}{\alpha^2\beta\gamma}+\frac{\mathrm{u}v^2\mathrm{w}}{\alpha\beta^2\gamma}+\frac{uvw^2}{\alpha\beta\gamma^2}\,.$$

Taking the inverse triple Natural transform of both sides we get

$$f(x,y,t)=x+y+t.$$

Example 3

Consider the linear volterra integro-differential equation

$$\frac{\partial^3 f(x,y,t)}{\partial x\partial y\partial t}+f(x,y,t)=x\,cost-\frac{x^2\,y\sin t}{2}+\int_0^x\int_0^y\int_0^t f(k,r,s)dkdrds, \qquad (4.9)$$

with initial condition

$f(x,y,0)=x,\ f(x,0,t)=x\,cost,\ f(0,y,t)=0,\ f(0,0,t)=0,\ f(x,0,0)=x,\ f(0,y,0)=0\,,$
$f(0,0,0)=0$ (4.10)

Solution.

Appling the triple Natural transform on both sides of (4.9), and double Natural transform of initial conditions equation (4.10), then we get

$$N_{+}^{3}\left[\frac{\partial^{3} f(x,y,t)}{\partial x\partial y\partial t}\right]+N_{+}^{3}\left[f(x,y,t)\right]=N_{+}^{3}\left[x\, cost-\frac{x^{2}\, y\sin t}{2}+\int_{0}^{x}\int_{0}^{y}\int_{0}^{t} f(k,r,s)dkdrds\right]$$

$$\begin{aligned}\frac{\alpha\beta\gamma}{uvw}\, N_{+}^{3}\left[f(x,y,t)\right]-\frac{\alpha\beta}{uv}\, N_{+}^{3}\left[f(x,y,0)\right]-\frac{\alpha\gamma}{uw}\, N_{+}^{3}\left[f(x,0,t)\right]-\frac{\beta\gamma}{vw}\, N_{+}^{3}\left[f(0,y,t)\right]&\\ +\frac{\alpha}{u}\, N_{+}^{3}\left[f(x,0,0)\right]+\frac{\beta}{v}\, N_{+}^{3}\left[f(0,y,0)\right]+\frac{\gamma}{w}\, N_{+}^{3}\left[f(0,0,t)\right]-\frac{N_{+}^{3}\left[f(0,0,0)\right]}{uvw}&\\ +N_{+}^{3}\left[f(x,y,t)\right]&\\ =\frac{u^{2}v\gamma}{\alpha^{2}\beta w}\left[\frac{1}{1+\frac{\gamma^{2}}{w^{2}}}\right]-\frac{u^{3}v^{2}}{\alpha^{3}\beta^{2}}\left[\frac{1}{1+\frac{\gamma^{2}}{w^{2}}}\right]+\frac{uvw}{\alpha\beta\gamma}\, N_{+}^{3}\left[f(x,y,t)\right],& \qquad (4.11)\end{aligned}$$

and

$$N_{+}^{3}\left[f(x,y,0)\right]=\frac{u^{2}\mathrm{v}}{\alpha^{2}\beta},\;\; N_{+}^{3}\left[f(x,0,t)\right]=\frac{u^{2}\gamma}{\alpha^{2}w}\left[\frac{1}{1+\frac{\gamma^{2}}{w^{2}}}\right],\; N_{+}^{3}\left[f(0,y,t)\right]\;=\;0$$

$$N_{+}^{3}\left[f(x,0,0)\right]=\frac{u^{2}}{\alpha^{2}},\;\; N_{+}^{3}\left[f(0,y,0)\right]=0\;,\; N_{+}^{3}\left[f(0,0,t)\right]=0,\;\; N_{+}^{3}\left[f(0,0,0)\right]=0\;. \qquad (4.12)$$

Substituting the equation (4.12) in the equation (4.11), then we have

$$\begin{aligned}\left(1-\frac{uvw}{\alpha\beta\gamma}+\frac{\alpha\beta\gamma}{uvw}\right)N_{+}^{3}\left[f(x,y,t)\right]&=\frac{u^{2}v\,\gamma}{\alpha^{2}\beta\, w}\left[\frac{1}{1+\frac{\gamma^{2}}{w^{2}}}\right]-\frac{u^{3}v^{2}}{\alpha^{3}\beta^{2}}\left[\frac{1}{1+\frac{\gamma^{2}}{w^{2}}}\right]+\frac{u\,\gamma^{2}}{\alpha\, w^{2}}\left[\frac{1}{1+\frac{\gamma^{2}}{w^{2}}}\right]\\ &=\left(\frac{u^{2}v\,\gamma}{\alpha^{2}\beta\, w}-\frac{u^{3}v^{2}}{\alpha^{3}\beta^{2}}+\frac{u\,\gamma^{2}}{\alpha\, w^{2}}\right)\left[\frac{1}{1+\frac{\gamma^{2}}{w^{2}}}\right]\\ &=\left(\frac{u^{2}v\,\gamma}{\alpha^{2}\beta\, w}\right)\left(1-\frac{uvw}{\alpha\beta\gamma}+\frac{\alpha\beta\gamma}{uvw}\right)\left[\frac{1}{1+\frac{\gamma^{2}}{w^{2}}}\right]\end{aligned}$$

$$N_{+}^{3}\left[f(x,y,t)\right]=\left(\frac{u^{2}v\,\gamma}{\alpha^{2}\beta\, w}\right)\left[\frac{1}{1+\frac{\gamma^{2}}{w^{2}}}\right]$$

Appling triple Natural transform of both sides, then we get

$$f(x,y,t)=x\; cost.$$

5. Conclusion

In this work we successfully applied the different properties of the triple natural transform to find the exact solutions of certain linear Volterra integro-differential equations in three dimensions subject to some initial conditions. The method was given in this paper is also applicable to the solution of problems in engineering, applied mathematics, physics and other fields where the researchers often search for the solutions of the LVIDEs arising in connection with their research problems.

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