Paths of Zeros in Analytic Representation of Finite Quantum Systems Employ Different Species of Matrices

Hend Abdelgader Eissa
Computing and Informatics Department
Electronic Technology college-Tripoli
namarek2010@gmail.com

Abstract

A quantum systems in $\mathscr{D}$ -dimensional Hilbert space Employ arithmetic technology. We concentrate on a set analytic representation in a cell $\mathfrak{S}$ which represents the finite quantum system. The time evolution of the system produces d paths of zeros. The main concept is to confine, our attention, to matrices such as: Swap matrix and vandemonde matrix get in matrix form instead periodic Hamiltonian matrix.The real power of the Displacement operator $\mathfrak{T}$, has periodic system as operator evolution time, is described. In numerical models in particular various, which demonstrate these motif, are introduced in this paper.

Keyword - Quantum system, Displacement operator, Swap matrix, vandemonde matrix.

I. Introduction

An analytic representation in the finite quantum systemhave been considered greatly in the literature (e.g., [1], [2], [3], [4]). which is describe their quantum states with analytic functions on a Torus, utilizing Theta functions. It mines that these functions has exactly d of zeros. In this paper we study the zeros of analytic representation illustrate finite quantum systems with variables in $\mathbb{Z}_d$. the paths of the zeros of periodic system using some important types of matrices are studied in details as : Swap matrix and vandemonde matrix. We find that every path is described by pluralism M,and by a pair of the winding number (w_1, w_2).Can also show that how four paths with pluralism $M = 1$, can join into one path with pluralism $M = 3$ (section IV). Displacement operators$\mathcal{Z}^{\alpha}\mathcal{X}^{\beta}$ are explained in finite quantum systems in the ring of $\alpha, \beta \in \mathbb{Z}(d)$, to analysis these operators to a real power $\mathfrak{T}$. It display that the paths of the zeros are exactly alike., but shifted, with respect to each other(section V). Several numerical example that illustrates these idea.The main goal in this paper is to develop the full quantum formalism in part of the zeros, and to get general laws that illustrate that shifting.

II. Finite quantum systems for Analytic representation

The position and the momentum take values in $\mathbb{Z}_d$ is introduced by a finite quantum system. It is studied with the d-dimensional Hilbert space $\mathcal{H}(d)$. In this case we Symbolize the position state and momentum state by $|X, n\rangle$and $|P, n\rangle$ (where the values of $n \in \mathbb{Z}_d$) which are related through a Fourier transform (FT), as follows is given by:

$$|P; n\rangle = F|X; n\rangle; \qquad F = d^{-1/2}\sum_{m,n}\omega(mn)|X; m\rangle\langle X; n| \tag{1}$$

where

$$\omega(m) = \exp\left[i\frac{2\pi m}{d}\right] \tag{2}$$

Let $|K\rangle$ be an arbitrary pure normalised state :

$$|\mathcal{K}\rangle = \sum_m \mathcal{K}_m|X; m\rangle; \qquad \sum_m |\mathcal{K}_m|^2 = 1 \tag{3}$$

We define the analytic representation of the state $|k\rangle$ as follows [6], [7],[10],[14].

$$\begin{aligned} K(Z) &= [\mathcal{N}(Z)]^{1/2}d^{1/2}\exp(\frac{-i}{2}Z_I Z_R)\langle Z^*|\mathcal{K}\rangle \\ &= \pi^{-1/4}\sum_{m=0}^{d-1}\mathcal{K}_m\,\Theta_3\left[\frac{\pi m}{d} - Z\sqrt{\frac{\pi}{2d}}; \frac{i}{d}\right] \end{aligned} \tag{4}$$

Where Θ_3 is the Theta function [17]

$$\begin{aligned} \Theta_3(u,\tau) &= \sum_{n=-\infty}^{\infty}\exp(i\pi\tau n^2 + i2nu); \\ \Theta_3'(u,\tau) = \frac{d\Theta_3}{du} &= i\sum_{n=-\infty}^{\infty}2n\exp(i\pi\tau n^2 + i2nu). \end{aligned} \tag{5}$$

The function $K(Z)$ is quasi-periodic.

$$\begin{aligned} K(Z + \sqrt{2\pi d}) &= K(Z) \\ K(Z + i\sqrt{2\pi d}) &= K(Z)\exp\left(\pi d - iZ\sqrt{2\pi d}\right), \end{aligned} \tag{6}$$

and subsequently it is sufficient to have this function in a cell S as.

$$S=[M\sqrt{2\pi d},(M+1)\sqrt{2\pi d})\times[N\sqrt{2\pi d},(N+1)\sqrt{2\pi d}) \tag{7}$$

where M and N are integers labelling the cell S. The analytic function $K(Z)$ has totally d zeros ζ_n in every cell S, [7], [10] and the following constraint.

$$\sum_{n=1}^{d}\zeta_n=\sqrt{2\pi d}(M+iN)+d^{3/2}\sqrt{\frac{\pi}{2}}(1+i). \tag{8}$$

In the finite system the $(d-1)$ zeros find uniquely the state (the last zero is defined from Equation.(8). Also in the infinite systems the zeros do not determine uniquely the state.

The last one can be found from Equation.(8),if the $(d-1)$ zeros ζ_n are given, and the function $K(Z)$ is given by

$$\begin{aligned}&K(Z)=\mathcal{N}(\{\zeta_n\})\\&\times\exp\left[-i\sqrt{\frac{2\pi}{d}}NZ\right]\prod_{n=1}^{d}\Theta_3\left[\sqrt{\frac{\pi}{2d}}(Z-\zeta_n)+\frac{\pi(1+i)}{2};\ i\right]\end{aligned} \tag{9}$$

Here N is the integer that labels the cell$S$ (as in Eq.(8), and $\mathcal{N}(\{\zeta_n\})$ is a constant determined by the normalization condition. Below we choose the cell with $M=N=0$.The proof of Eq(8),(9) are giver in ([13],[14]).

III. Paths of zeros and time evolution

Let $\mathcal{L}$ be a matrices of the system (a $d\times d$ Hermitian matrix L_{mv}). As the system develops in time $\mathfrak{T}$, every zero ζ_n follows a path $\zeta_n(\mathfrak{T})$. In this case the actual path using matrices with different properties are studied and calculate the analytic expressions for the derivatives of the function $\zeta_n(k_0,...,k_{d-1})$ in numerical program. Therefore we using the following equation to plot the graphs:

$$\zeta_n+\Delta\zeta_n=\zeta_n+\sum_m\frac{\partial\zeta_n}{\partial k_m}\Delta k_m=\zeta_n+i\Delta t\sum_{m,v}\frac{\partial\zeta_n}{\partial k_m}L_{mv}k_v. \tag{10}$$

Where,

$$\frac{\partial\zeta_n}{\partial k_m}=-\frac{\pi^{-1/4}\Theta_3\left[\frac{\pi m}{d}-\zeta_n\sqrt{\frac{\pi}{2d}};\frac{i}{d}\right]}{\mathcal{N}(\{\zeta_n\})\sqrt{\frac{\pi}{2d}}A_n(\zeta_n)\Theta_3'\left[\frac{\pi(1+i)}{2};\ i\right]}. \tag{11}$$

Here

$$\Theta_3'\left[\frac{\pi(1+i)}{2};\ i\right]=1.99i. \tag{12}$$

In each step of the repeat process $\mathcal{N}(\{\zeta_n\})$ is calculated as

$$\mathcal{N}(\{\zeta_n\})=\frac{\pi^{-1/4}\sum_{m=0}^{d-1}k_m\ \Theta_3\left[\frac{\pi m}{d}-Z\sqrt{\frac{\pi}{2d}};\frac{i}{d}\right]}{\prod_{n=1}^{d}\Theta_3\left[\sqrt{\frac{\pi}{2d}}(Z-\zeta_n)+\frac{\pi(1+i)}{2};\ i\right]} \tag{13}$$

As mentioned above, the $\mathcal{N}(\{\zeta_n\})$ dose independent on Z and any value of it can be employed for its numerical computation

IV. Numerical Example using Various Types of Matrices

We consider periodic systems with matrices such that
$\exp(iT\mathcal{L})$=1 for some T. This happen when the ration of the eigenvalues of $\mathcal{L}$ are rational numbers. In here results base on the analytic method illustrated in (section II). As example we consider the Swap matrices:

$$S=\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix} \tag{14}$$

In this case the period$T$ is equal 2π. It can assumes at $t=0$ the zeros are given in the following equation:

$$\begin{aligned}&\zeta_0(0)=1+2.02i;\quad\zeta_1(0)=4.02+3.5i;\\&\zeta_2(0)=1+1.5i;\quad\ \ \zeta_3(0)=4+3i.\end{aligned} \tag{15}$$

Using Eq.(10) It can generate the paths of the zeros. Results are shown in Figs.1. In this case there are one path with pluralism $M=1$ and the other paths with pluralism $M=3$. We note that:

$$\begin{aligned}&\zeta_0(T)=\zeta_0(0);\quad\zeta_1(T)=\zeta_2(0);\\&\zeta_2(T)=\zeta_3(0);\quad\zeta_3(T)=\zeta_1(0).\end{aligned} \tag{16}$$

And also consider the Swap matrix in Eq.(14) then suppose that at $t=0$ the zeros are the following:

$$\begin{aligned}&\zeta_0(0)=2+2.02i;\quad\zeta_1(0)=2.02+3i;\\&\zeta_2(0)=1+4i;\qquad\ \zeta_3(0)=5-T+1i;\end{aligned} \tag{17}$$

In this case the results are given in Fig.2. can see that there are four paths with pluralism $M = 1$.

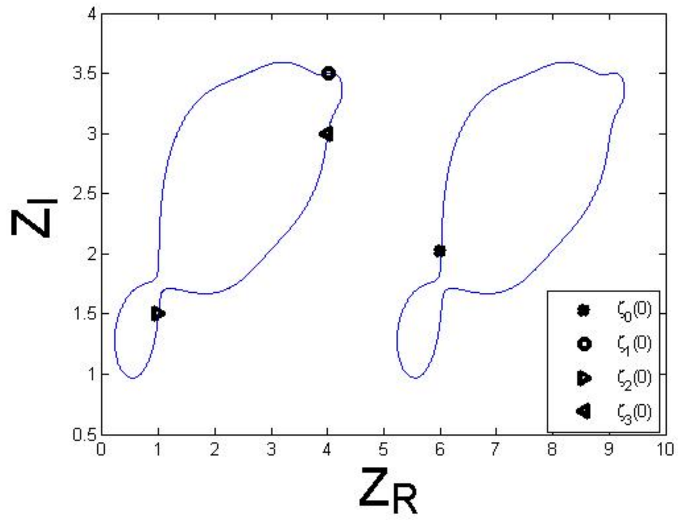

Fig. 1. Paths of the zeros for the Swap of Eq.(14). At $t = 0$ the zeros are given in Eq.(15)

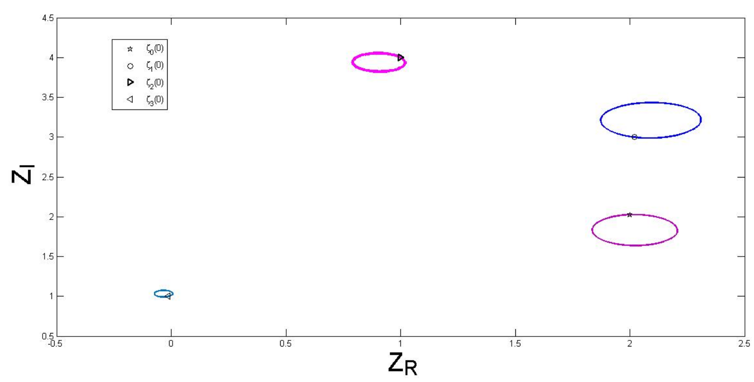

Fig. 2. Paths of the zeros for the Swap of Eq.(14). At $t = 0$ the zeros are given in Eq.(17)

Know consider the vandermonde matrix as following:

$$S = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \tag{18}$$

The period is $T = 3\pi$. then suppose that at $t = 0$ the zeros are the following:

$$\begin{aligned} \zeta_0(0) &= 1 + 3.02i; \quad \zeta_1(0) = 3.02 + 3i; \\ \zeta_2(0) &= 1 + 3i; \quad \zeta_3(0) = 5 + 1i; \end{aligned} \tag{19}$$

The paths of zeros are given in Fig.3.The winding numbers (w_1, w_2) of the four paths are $(0, 0)$, $(0, 0)$,$(4, 0)$ and $(4, 0)$.

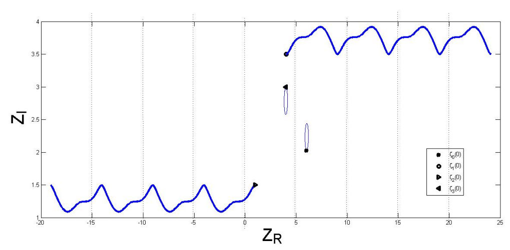

Fig. 3. Paths of the zeros for the Swap of Eq.(18). The period is $T = 3\pi$ At $t = 0$ the zeros are given in Eq.(19). Dotted line show a cell which is square wit each side equal to 5.01

V. The analytic representation of Zeros of $X^{\mathfrak{T}}|K\rangle$

In The phase space($\mathbb{Z}(d) \times \mathbb{Z}(d)$) the Displacement operators defined as:

$$\mathcal{Z} = \sum_n \omega(n)|X;n\rangle\langle X;n| = \sum_n |P;n+1\rangle\langle P;n|$$
$$\mathcal{X} = \sum_n \omega(-n)|P;n\rangle\langle P;n| = \sum_n |X;n+1\rangle\langle X;n|$$
$$\mathcal{X}^d = \mathcal{Z}^d = \mathbf{1}; \qquad \mathcal{X}^\beta \mathcal{Z}^\alpha = \mathcal{Z}^\alpha \mathcal{X}^\beta \omega(-\alpha\beta); \tag{20}$$

Here $\alpha, \beta \in \mathbb{Z}(d)$. In this paper the zeros of the analytic representation of $\mathcal{X}^{\mathfrak{T}}|k\rangle$ are studied which defined as $K(z;\mathfrak{T})$, here $\mathfrak{T} \in \mathbb{R}$. The operator $\mathcal{X}^{\mathfrak{T}}$ can be considered as a time evolution operator $\exp(i\mathfrak{T}\mathcal{L})$ with Hamiltonian $\mathcal{L} = -i \ln \mathcal{X}$ (we note here the logarithm is multi-valued and we take the principal value).

Let $|k\rangle = \sum \widetilde{k}_m |P;m\rangle$, here $\widetilde{k}_m$ be the Fourier transforms of the k_m in Eq.(3).
In this context the state $\mathcal{X}^{\mathfrak{T}}|k\rangle = \sum [\omega(-m)]^{\mathfrak{T}} \widetilde{k}_m |P;m\rangle$ is represented by the function $K(z,\mathfrak{T})$ as:

$$K(Z,\mathfrak{T}) = \pi^{-1/4} \exp\left(-\frac{Z^2}{2}\right) \sum_{m=0}^{d-1} \exp\left(-\frac{i2\pi m \mathfrak{T}}{d}\right) \widetilde{k}_m \times \Theta_3\left[\frac{\pi m}{d} - iZ\sqrt{\frac{\pi}{2d}}; \frac{i}{d}\right] \tag{21}$$

We represented the $|P;m\rangle$ by $\pi^{-1/4} \exp\left(-\frac{Z^2}{2}\right) \Theta_3\left[\frac{\pi m}{d} - iz\sqrt{\frac{\pi}{2d}}; \frac{i}{d}\right]$[12]. The zeros of $G(Z;\mathfrak{T})$ here is $\zeta_n(\mathfrak{T})$ i.e.,

$$G[\zeta_n(\mathfrak{T});\mathfrak{T}] = 0 \tag{22}$$

where $n = 0, ..., d-1$. In this case we can see in Fig.4 and Fig.5 every path of zeros $\zeta_{n+\beta}$, is a moved conversion (in one direction) of another path ζ_n. The position of the zero on every path at a certain time, is the same position of the zero on another path, at a different time:

$$\zeta_{n+\beta}(\mathfrak{T} + \beta) = \zeta_n(\mathfrak{T}) + \beta\sqrt{\frac{2\pi}{d}}; \quad n, \beta \in \mathbb{Z}(d). \tag{23}$$

In this part of the paper it draw the zeros and there paths of this state $\mathcal{X}^{\mathfrak{T}}|k\rangle$. The state $|k\rangle$ which defined from the zeros at period time $\mathfrak{T} = 0$ given in Fig (4):

$$D = \begin{bmatrix} 0 & -0.4898 - 0.8717i & 0 \\ 0 & 0 & 1.0000 - 0.0121i \\ -0.5000 + 0.8660i & 0 & 0 \end{bmatrix} \tag{24}$$

then suppose that the zeros at $\mathfrak{T} = 0$ are the following:

$$\zeta_0(0) = 2.1710 + 0.7210i; \ \zeta_1(0) = 2.1710 + 2.1710i; \ ; \zeta_2(0) = 2.1704 + 3.6204i \tag{25}$$

It can see that their are d smiler paths, which are moved in both of direction (z_R and z_I)by $\sqrt{2\pi/d}$. In Fig 5 it plots the paths of the zeros of the function Eq (21) which represented the state $D^{\mathfrak{T}}|k\rangle$ as following:

$$D = \begin{bmatrix} 0 & 0 & 0.9946 + 0.0086i \\ -0.4843 + 0.8690i & 0 & 0 \\ 0 & -0.5000 - 0.8600i & 0 \end{bmatrix} \tag{26}$$

It can see that every paths has one pluralism $M = 1$ and the full path of zeros through a period of T, which is moved of the path to another zero.

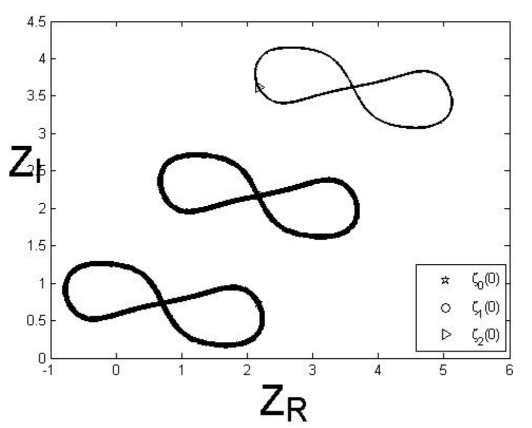

Fig. 4. The paths of the zeros of the state $[\mathcal{D}]^{\mathfrak{T}}|k\rangle$ given in Eq 26. The state $|k\rangle$ is defined over the zeros at $\mathfrak{T} = 0$ shown in Eq.(25)

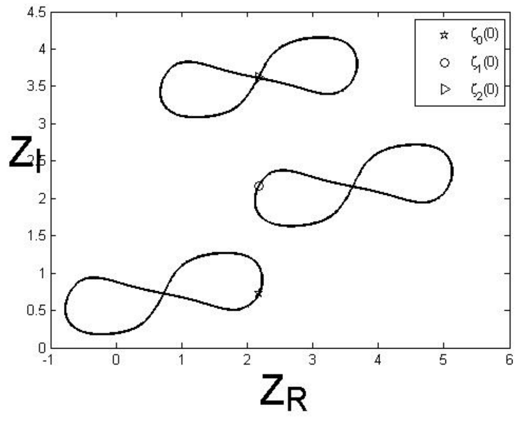

Fig. 5. The paths of the zeros of the state $[\mathcal{D}]^{\mathfrak{T}}|k\rangle$ given in Eq 26. The state $|k\rangle$ is defined over the zeros at $\mathfrak{T} = 0$ shown in Eq.(25)

VI. Discussion

In this paper we have investigated Finite Hilbert space with quantum systems for values of d, where the positions and the momenta state takes values in $\mathbb{Z}(d)$. The Theta function are used in An analytic representation, which describes these systems, has been shown in Eq.(4). The d zeros of these analytic functions determine uniquely the state of the system.

Numerical method used to calculate these paths of the zeros, has been given in Eqs.(10),(11). every path is characterized by the pluralism M (discussed in section IV), and by a pair of winding numbers (w_1, w_2)(discussed in section IV). Furthermore, we studied the periodic system which has the displacement operator to the real power $\mathfrak{T}$, as time evolution operator $\mathcal{X}^{\mathfrak{T}}|K\rangle$(discussed in section V). The work provides a deeper understanding of systems with finite Hilbert space, through their analytic representation and in particular through its zeros.

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