A comparative study of some numerical methods for solving first order differential equation.

by

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Abstract

In many cases fail to the previous method or others fail to solving the first order differential equation a closed or complete solution, and in such cases, there is no way to avoid going to the approximate solutions.

Numerical methods occupy a prominent position among the approximate methods of solving in the field of computers.

Keywords

Euler's method, Heun's method, Taylor's series method, the Range Cotta method, compare, closed solution and approximate solution.

.*Introduction*

In this paper, we discuss a comparative study between four numerical methods for solving a first order differential equation and knowing the least error between the complete or closed solution and approximate solutions of the four methods.

It is considered a part that includes basic mathematical operations which lead to an approximate solution that fulfills certain conditions.

If the problem of the initial value of the first order is in the following explicit form:

$$\frac{dy}{dx} = f\,(x,y);\ y\,(x_0) = y_0 .1.1$$

Where $f\,(x, y)$ is generally a unique valued function in x and y at all points of R .

Avoid to use but it may be a function of x or y only, in x and y.

Any numerical method enables us to find approximate solutions$x_1, x_2, x_3, \ldots$

Where, the difference between any two successive values of x is a constant quantityh called the step and we choose it appropriately, in other words it increases as an arithmetic sequence whose base ish,where his:

$$\boldsymbol{h = x_{n+1} - x_n \ ; \quad n = 0, 1, \ldots 1.2}$$

We symbolize the approximate solution corresponding to the value of x_n with the symbol $\tilde{y}_n (x)$ and the corresponding perfect solution is symbolized by the symbol $y_n(x)$ meaning that $\tilde{y}_n = y_n$ and the error $y_n - \tilde{y}_n$ decreases as the difference decreasesh $= x_{n+1} - x_n$.

Given x_n , $\tilde{y}_n$ the approximate derivative at this point can be calculated from the following equation.

$$\boldsymbol{\left(\frac{\widetilde{dy}}{dx}\right)_n = \tilde{y}'_n = f(x_n \,; \tilde{y}_n). 1.3}$$

We now review a comparison of the several numerical methods:

I- Euler's method(step - by – stepmethod)

At the point (x_n , $\tilde{y}_n$) ,the value of the derivative is approx to:

$$\boldsymbol{\tilde{y}'_n = \frac{\tilde{y}_{n+1} - \tilde{y}_n}{x_{n+1} - x_n} = \frac{1}{h}(\tilde{y}_{n+1} - \tilde{y}_n)}$$

Substituting *(1.3)* for $\tilde{y}_n$, we get.

$$\boldsymbol{\tilde{y}_{n+1} = \tilde{y}_n + hf(x_n, \tilde{y}_n) 1.4}$$

Or

$$\boldsymbol{\tilde{y}_{(x_{n+1})} = \tilde{y}_{(x_n)} + hf(x_n + \tilde{y}_n).}$$

We start the solution from the initial point (x_0 , y_0) located on the required integral curve and then we solve $y(x)$ to get the approximate point $x_1 = x_0 + h$, $\tilde{y}_1$

That is on the approximate value $\tilde{y}_1$ from the relation (1.4) by setting $n = 0$.

$$\boldsymbol{\tilde{y}_1 = y_0 + hf(x_0 \,; y_0) 1.5}$$

This is because $\tilde{y}_0 = y_0$ from the approximate point ($x_1, \tilde{y}_1$) as the following(A.1), by taking the horizontal components of the equal distances, $\Delta x_1 = \Delta x_2 = \cdots = h$, the equation (1.4)means that the integral curve $y(x)$ in the interval $[x_{n\,;}\, x_n + h]$ to a straight line passing through the point ($y_{n\,,} \tilde{y}_n$).

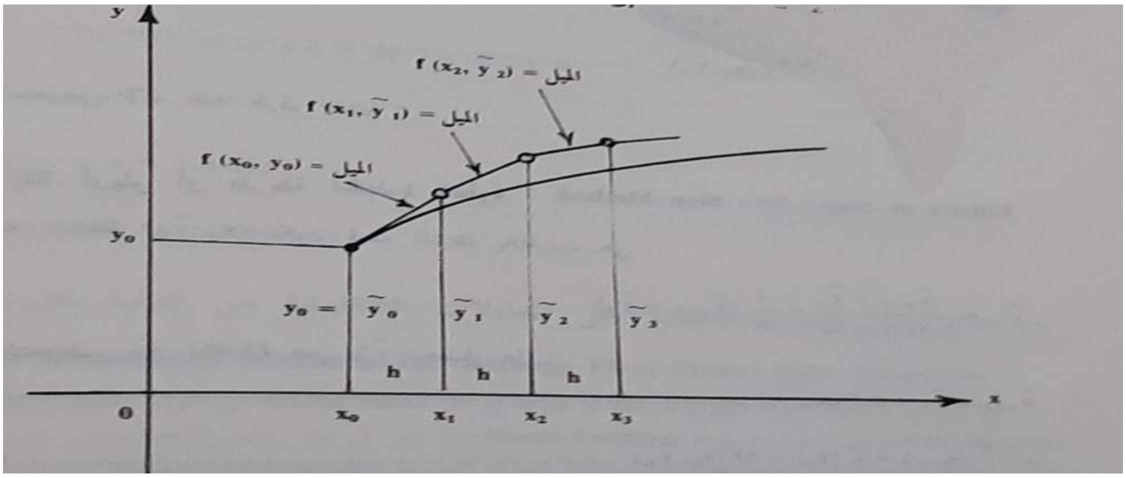

Figure A. 1 Euler's method (steps by step)

We take an illustrative example of Euler's method in order to test the accuracy of this method.

Example

Solve the following an initial value differential equationby using Euler's method?

$$\frac{\partial y}{\partial x}+2y=10 \quad ; \quad y(0)=0 . \qquad *$$

To find $y(x)$,assuming that $h = 0.2$, solve the problem as a closed or complete solution and compare the approximate solution in both cases with the complete solution.

Solution

The given equation is a linear differential equation

First, we find the integral factor

$$\mu(x) = e^{\int 2dx} = e^{2x}$$

Multiplying the integral factor by equation (*), we find that:

$$e^{2x}y = 5e^{2x} + c$$

at initial point $y(0) = 0$, we find that $c = -5$

$$y(x) = 5(1-e^{-2x}) \qquad **$$

This is the particular solution to the given equation.

To use Tayler's method, we put the equation on the form (1.1)

$$dy/dx = 2(5 - y) = f(x, y)$$

This implies that

$$f(x, y) = 2(5 - y)$$

Since the starting point is$(0, 0), h = 0.2$, then

$$\tilde{y}_1 = \tilde{y}_n = \tilde{y}(x_0 + nh)$$

Implies that ***nh=1***

Then $$n = \frac{1}{h} = \frac{1}{0.2} = 5$$

Which is required is to calculate $\tilde{y}_5$ from the equation *1.4.*

$$\tilde{y}_{n+1} = \tilde{y}_n + hf(x_n; \tilde{y}_n) \text{ ; } n = 0; 1; 2; 3; 4$$

At *n = 0,*

$$f(x_0, y_0) = 2(5 - 0) = 10$$

then $$\tilde{y}_1 = y_0 + hf(x_0; y_0) = 0 + 0.2(10) = 2$$

By similar calculation we find that:

At $n = 1$, $y_2 = 3.2$, $n = 2$, $y_3 = 3.29$, $n = 3$, $y_4 = 4.35$ and finally at $n = 4$, $y_5 = 4.6112$

We now calculate the exact value of $y(1.0)$ from the particular solution *(**)* rounded to four decimal places

y_5 = y (1.0) = 5(1- e^(-2)) = 5(0.8646)= 4.3233

The error between the two cases is

$$\Delta = \textit{Error} = |y_5 - \tilde{y}_5| = |4.3233 - 4.6112| = |0.2879|$$

The following table (1. a) combines the arithmetic steps for cases $h = 0.2$ and both the exact value and the absolute error.

n	x_n	$\tilde{y}_n$	$\tilde{y}'_n = f(x_n; \tilde{y}_n)$	$y_n = 5(1 - e^{-2x})$	$\Delta = \lvert y_n - \tilde{y}_n \rvert$
0	0.0	0.0000	10.0000	0.0000	0.0000
1	0.2	2.0000	6.0000	1.6484	0.3516
2	0.4	3.2000	3.6000	2.7534	0.4466
3	0.6	3.9200	2.1600	3.4940	0.4260
4	0.8	4.3520	1.2960	3.9905	0.3615
5	1.0	4.6112	0.7776	4.3233	0.2879

Table (1. a)

II) Heun's method

In this method, we first calculate the auxiliary value $\tilde{y}$

$$\tilde{\boldsymbol{y}}_{n+1} = \tilde{\boldsymbol{y}}_n + \boldsymbol{hf}(\boldsymbol{x}_n\,; \tilde{\boldsymbol{y}}_n)\boldsymbol{1.5}$$

Then we calculate the approximate value $\tilde{y}_{n+1}$ of

$$\tilde{\boldsymbol{y}}_{n+1} = \tilde{\boldsymbol{y}}_n + \frac{\boldsymbol{h}}{2}[\boldsymbol{hf}\ (\boldsymbol{x}_n\,, \tilde{\boldsymbol{y}}_n) + \boldsymbol{f}(\boldsymbol{x}_{n+1}, \tilde{\boldsymbol{y}}_{n+1})]\boldsymbol{1.6}$$

This improvement means that instead of approximating the integral curve $y(x)$ in the interval $[x_{n,} x_n + h]$ with a segment of one straight line with slope $f(x_n, \tilde{y}_n)$ and passing through point (x_n, y_n) ,it is approximated in the interval $\left[x_n, x_n + \frac{1}{2}h\right]$ by a segment of a straight line with slope $f(x_n, \tilde{y}_n)$ and through point $(x_n, \tilde{y}_n)$, and then in the rest of the interval $\left[x_n, x_n + \frac{1}{2}h\right]$ it is approximated by a second straight line segment with slope $(x_{n+1}, \tilde{y}_{n+1})$ it starts from the end of the first straight line segment as shown in the following figure (1 . B).

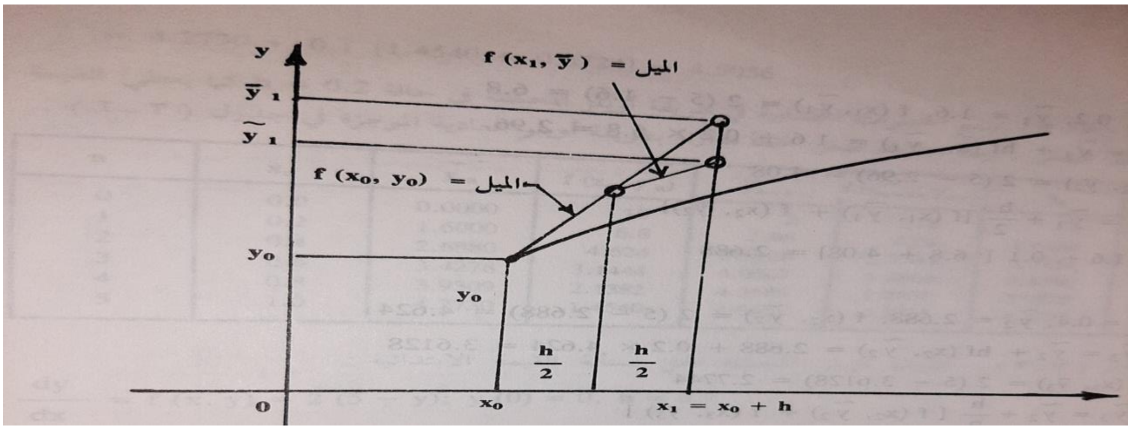

Figure (1. B)

We use the Heuns method to solve the previous example at $h = 0\,; n = 5$ of the equation

$$\frac{\boldsymbol{dy}}{\boldsymbol{dx}} = \boldsymbol{f}(\boldsymbol{x}; \boldsymbol{y})\ \ \boldsymbol{2}(\boldsymbol{5} - \boldsymbol{y}) \qquad\qquad ;\ \ \boldsymbol{y}(\boldsymbol{0}) = \boldsymbol{0}$$

From the equation (1.6)we find that:

At *n =0:*

$$\boldsymbol{f}(\boldsymbol{x}_0\,; \boldsymbol{y}_0) = \boldsymbol{2}(\boldsymbol{5} - \boldsymbol{0}) = \boldsymbol{10}$$

$$\tilde{\boldsymbol{y}}_1 = \boldsymbol{y}_0 + \boldsymbol{hf}(\boldsymbol{x}_0\,; \tilde{\boldsymbol{y}}_0) = \boldsymbol{(0 + 0.2)\ 10 = 2}$$

$$\tilde{y}_{n+1} = \tilde{y}_n + \frac{h}{2}[f(x_n\,;\tilde{y}_n) + f(x_{n+1}\,;\,\tilde{y}_{n+1})]$$

$$\tilde{y}_1 = y_0 + \frac{0.2}{2}[10 + 6] = 0.1(16) = 1.6$$

By similar calculate we find that:

At ***n=1,*** $\tilde{y}_2 = 2.688$, ***n =2***, $\tilde{y}_3 = 3.4278$,***n =3***, $\tilde{y}_4 = 3.9309$ and finally at ***n = 4,*** $\tilde{y}_5 = 4.2730$

We calculate the particular solution at $y(1.0)$ from the complete solution (**) and we find that:

$$y_5 = y(1.0) = 5(1 - e^{-2}) = 4.3233$$

Comparing the complete solution with the approximate solution, we find that the error or difference is:

∆ = error =|y_5-y ̃_5 | = |4.3233-4.2730|= 0.0503

We note that the error is small compared to the summary Euler method in the table (a. 1)

In the following table (1.b) the computational steps are grouped in the Heuns method at $h = 0.2$.

n	x_n	$\tilde{y}_n$	$f(x_n;\tilde{y}_n)$	$\tilde{y}_{n+1}$	$f(x_{n+1};\tilde{y}_{n+1})$	y_n	$\Delta = \lvert y_n - \tilde{y}_n \rvert$
0	*0.0*	*0*	*10*	*2*	*6*	*0*	*0*
1	*0.2*	*1.6*	*6.8*	*2.96*	*4.08*	*1.6484*	*0.0484*
2	*0.4*	*2.688*	*4.624*	*3.6128*	*2.7744*	*2.7534*	*0.0654*
3	*0.6*	*3.4278*	*3.1444*	*4.0567*	*1.8866*	*3.4940*	*0.0662*
4	*0.8*	*3.9309*	*2.1382*	*4.3585*	*1.2830*	*3.4905*	*0.0596*
5	*1.0*	*4.2730*	*1.4638*	*4.5638*	*0.8724*	*4.3233*	*0.0503*

Figure (1.b)

III) Taylor's series method

In the problem of an initial value $y'(x) = f(x,y)$; $y(x_0) = y_0$; it is possible by iterating differential to find the higher derivatives $y''_{(x)}, y'''_{(x)} \cdots$

Then we calculate the values of these derivatives at any value of $x_n = x_0 + nh$, we start by calculating $y(x)$ and its derivatives at the primitive point x_0 and then from Tayler'sseries, wecalculate that the solution at point $x_1 = x_0 + h$ is*:*

$$y(x_1) = y(x_0 + h) = y(x_0) + hy'^{(x_0)} + \frac{h^2}{2!}y''^{(x_0)} + \frac{h^3}{3!}y'''^{(x_0)} + \cdots$$

Or

$$\boldsymbol{y_1 = y_0 + hy_0{}' + \frac{h^2}{2!}y_0{}'' + \frac{h^3}{3!}y_0{}'''} \qquad 1.7$$

We can calculate all of y_2 ; y_3 ; ...

$$\boldsymbol{y_2 = y_1 + hy_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_3'''} \qquad 1.8 \qquad .$$

$$\boldsymbol{y_{n+1} = y_n + hy_n{}' + \frac{h^2}{2}y_n{}'' + \frac{h^3}{3}y_3{}'''} \qquad 1.9$$

The above equation gives the value y_{n+1} at x_{n+1} in terms of the values of y and their successive derivatives at the previous point x_0 .

It suffices with a reasonable number of limits according to the value of the differenceh $= x_{n+1} - x_n$.

We solve the previous example by using Tayler's method up to the fourth derivative at *h = 0.2* from the following given equation.

$$\boldsymbol{\frac{dy}{dx} = 2(5 - y)}$$

$$\left.\frac{dy}{dx}\right|_{x=0} = y'(0) = 2(5 - 0) = 10$$

We find the higher derivatives at the starting point (0, 0)

$$\left.\frac{d^2y}{dx^2}\right|_{x=0} = -2y_0' = -4(5 - y_0) = -20$$

$$\left.\frac{d^3y}{dx^3}\right|_{x=0} = -2y_0'' = 8(5 - y_0) = 40$$

$$\left.\frac{d^4y}{dx^4}\right|_{x=0} = -2y_0''' = -16(5 - y_0) = -80$$

From equation (1.9) where, $n = 0; 1; 2; 3; 4$, we find that:

At ***n= 0***

$$\boldsymbol{\tilde{y}_1 = y_0 + hy_0' + \frac{h^2}{2!}y_0''' + \frac{h^3}{3!}y_0''' + \frac{h^4}{4!}y_0^4}$$

$$\tilde{y}_1 = 0 + 0.2 \times 10 + \frac{(0.2)^2}{2}(-20) + \frac{(0.2)^3}{6}(40) + \frac{(0.2)^4}{24}(-80)$$

$\tilde{y}_1$ = ***1.6480***

$\tilde{y}_1'$ = ***2(5 -*** $\tilde{y}_1$***) = 2(5 – 1.648) = 6.704***

$$\boldsymbol{\tilde{y}_1'' = -2\tilde{y}_1' = -2(6.704) = -13.408}$$

$$\boldsymbol{\tilde{y}_1''' = -2\tilde{y}_1'' = -2(-13.408) = 26.816}$$

$$\tilde{y}_1^{(4)} = -2\tilde{y}_1''' = -2(26.816) = -53.632$$

And similar

$\boldsymbol{at\ \ n = 1}$

$\tilde{y}_2 = 2.7528$, $\tilde{y}_2' = 4.4944$, $\tilde{y}_2'' = -8.9887$,$\tilde{y}_2''' = 17.977$ *and* $\tilde{y}_2^{(4)} = -35.9548$

$at\ n = 2,$

$\tilde{y}_3 = 3.4935$, $\tilde{y}_3' = 3.013$, $\tilde{y}_3'' = -6.026$, $\tilde{y}_3''' = 12.052$ *and* $\tilde{y}_3^{(4)} = -24.104$

at n = 3,

$\tilde{y}_4 = 3.990$, $\tilde{y}_4' = 2.0199$, $\tilde{y}_4'' = -4.0398$,$\tilde{y}_4''' = 8.0796$, and$\tilde{y}_4^{(4)} = -16.159$

And at $n = 4$,$\tilde{y}_5 = 4.3229$

Comparing the complete solution with the approximate solution, we find that the error or difference is

$$\Delta = \boldsymbol{error} = |\boldsymbol{y_5} - \boldsymbol{\tilde{y}_5}| = |\boldsymbol{4.3233 - 4.3229}| = \ |\boldsymbol{0.0004}|$$

So, the difference or error is $\Delta = 0.0004$.

We note that the error is small compared to the Euler's method and the Heun's method.

The following table (1.c) shows the arithmetic steps in the Tayler's method up to the fourth derivative in at *h = 0.2.*

n	x_n	$\tilde{y}_n$	$\tilde{y}_n'$	$\tilde{y}_n''$	$\tilde{y}_n'''$	$\tilde{y}_n^{(4)}$	y_n	$\Delta = \lvert y_n - \tilde{y}_n \rvert$
0	0.0	0.0000	10.000	-20.000	40.000	-80.000	0.0000	0.0000
1	0.2	1.6480	6.7040	-13.408	26.816	-53.632	1.6484	0.0004
2	0.4	2.7528	4.4944	-8.9887	17.9774	-35.9549	2.7534	0.0006
3	0.6	3.4935	3.0130	-6.0260	12.052	-24.104	3.4940	0.0005
4	0.8	3.9900	2.0199	-4.0398	8.0796	-16.1592	3.9905	0.0005
5	1.0	4.3229	1.3542	-2.7084	5.4168	-10.8336	4.3233	0.0004

Table (1.c)

IV) Runge –Kutta Method

This method goes back to the German mathematicians Carl-Runge (1856/1927) and Wilhelm Cutta (1867/1944).

To solve an initial value problem

$$\boldsymbol{\frac{dy}{dx} = f(x,y);\ y(x_0){=}y_0}\ ,$$

We first calculate the following four quantities using **the Runge-Kutta method**

$$A_n = hf(x_n; \tilde{y}_n) \qquad 1.10$$

$$B_n = hf(x_n + \tfrac{1}{2}h\,;\ \tilde{y}_n + \tfrac{1}{2}B_n) \qquad 1.11$$

$$C_n = hf\left(x_n + \tfrac{1}{2}h\,;\ \tilde{y}_n + \tfrac{1}{2}B_n\right) \qquad 1.12$$

$$D_n = hf(x_n + h\,;\ \tilde{y}_n + C_n) \qquad 1.13$$

Then we calculate the value $\tilde{y}_{n+1}\ \ atx_n + h$ from the equation

$$\tilde{y}_{n+1} = \tilde{y}_n + \tfrac{1}{6}(A_n + 2B_n + 2C_n + D_n) \qquad 1.14$$

It can be shown that the error in **the Runge Kutta method** is very small within the limits h at $n = 5$

Where $h = x_{n+1} - x_n$

The equation to be solved using the Runge-Kutta method to find y(1.1) is

$$\frac{dy}{dx} = f(x; y) = 2(5 - y) \quad ; \ y(0) = 0$$

Where $h = 0.2$,so we apply the method to values $n = 0,1,2,3,4$

We calculate that the auxiliary quantities from the equations (1.10) ,(1.11) ,(1.12) ,(1.13).

At n = 0,

$$A_0 = hf(x_0\,;\ y_0) = hf(0; 0) = 0.2\ (10 - 0) = 2$$

$$B_0 = hf\left(x_0 + \frac{1}{2}h\,; y_0 + \frac{1}{2}A_0\right) = 0.2x2(5 - 1) = 1.6$$

$$C_0 = hf\left(x_0 + \frac{1}{2}h\,;\ y_0 + \frac{1}{2}B_0\right) = 0.2x2(5 - 0.8) = 1.68$$

$$D_0 = hf\left(x_0 + \frac{1}{2}h\,; y_0 + \frac{1}{2}C_0\right) = 0.2x2(5 - 0.84) = 1.328$$

From equation (1.14), we find that $n = 0$

$$\tilde{y}_1 = y_0 + \frac{1}{6}(A_0 + 2B_0 + 2C_0 + D_0)$$

$$\tilde{y}_1 = 0 + \frac{1}{6}(2 + 3.2 + 3.36 + 1.328) = 1.6480$$

By similar calculation we find that:

At n = 1,

$A_1 = 1.3408$,$B_1 = 1.0726$,$C_1 = 1.1263$, $D_1 = 0.8903$,

implies that $\tilde{y}_2 = 2.7528$

,$n = 2$, $A_2 = 0.8989$, $B_2 = 0.719$,$C_2 = 0.7550$ *and* $D_2 = 0.5969$,

implies that $\tilde{y}_3 = 3.4935$

,$n = 3$, $A_3 = 0.6026$, $B_3 = 0.4821$, $C_3 = 0.5062$ *and* $D_3 = 0.4001$,

implies that $\tilde{y}_4 = 3.9901$

and n = 4,$A_4 = 0.4039$,$B_4 = 0.3232$, $C_4 = 0.3393$ *and* $D_4 = 0.2682$,

implies that $\tilde{y}_5 = 4.3230$

We calculate that the particular solution at $y(1.0)$ from the general solution (**)

$$y_5 = y(1.0) = 5(1 - e^{-2}) = 4.3233$$

So, the error or difference between the closed solution and general solution in this method is

$$\Delta = |y_5 - \tilde{y}_5| = |4.3233 - 4.3230| = |0.0003|$$

Table (1.d) shows the commutation steps of the **Runge-Kutta method**

N	x_n	$\tilde{y}_n$	A_n	B_n	C_n	D_n	y_n	$\Delta = \lvert y_n - \tilde{y}_n \rvert$
0	*0*	*0.0000*	*2.0000*	*1.6000*	*1.6800*	*1.3280*	*0.0000*	*0.0000*
1	*0.2*	*1.6480*	*1.3408*	*1.0726*	*1.1263*	*0.8903*	*1.6484*	*0.0004*
2	*0.4*	*2.7528*	*0.8989*	*0.7191*	*0.7551*	*0.5969*	*2.7534*	*0.0006*
3	*0.6*	*3.4935*	*0.6026*	*0.4821*	*0.5062*	*0.4001*	*3.4940*	*0.0005*
4	*0.8*	*3.9901*	*0.4040*	*0.3232*	*0.3393*	*0.2682*	*3.9905*	*0.0004*
5	*1.0*	*4.3230*					*4.3233*	*0.0003*

Table (1.d)

Conclusion

We conclude from this research that the **Range Cotta method** is more accurate in calculating between the closed solution and the approximate solution than the other three methods, which are**Euler's method**, **Heun's method** and **Taylor series**,however the **Taylor series method** is not far from the accuracy of the solution from **the Runge Kutta method,** but this method is defective in its dependenceon the calculation of higher derivatives, which have some complexity in the solution

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