An Overview of Feynman's Method for Calculating Improper Integrals

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ABSTRACT

The objective of this paper is to utilize the Feynman method to find the definite integral of some functions that do not have an elementary antiderivatives. The main idea behind "Feynman's method" is to solve a simple differential equation instead of computing an integral. In most of the examples have been presented, we concentrated on uncommon integrations to demonstrate the efficiency of this method.

Keywords: Feynman's method, differentiation under the integral sign rule ,definite integral and improper integral.

1 Introduction

The Fundamental Theorem of Calculus cannot be used to evaluate the definite integrals of many continuous functions because their antiderivatives lack basic formulas. Numerical integration provides a practical approach to estimating the values of these so-called "non-elementary integrals" as long as there is a procedure (or program code) in your computer or calculator for numerical integration (Stewart, Clegg, & Watson, 2020).

In calculus, every elementary function-based single-variable function has a derivative. However, not every elementary function-based single-variable function can be exactly integrated over any arbitrary integral; for instance, the antiderivative cannot be expressed in terms of elementary functions, in order to find the exact integral for any interval. *Elliptic integral*, *logarithmic integral*, *error function*, *Gaussian integral*, *Fresnel integral sine integral* (*Dirichlet integral)*, *Exponential integral* (*in terms of the exponential integral*) and *logarithmic integral* (*in terms of the logarithmic integral)* are examples of functions with non-elementary antiderivatives (Zwillinger, D. and Jeffrey, A., 2007) and (Jeffrey, A. and Dai, H.H., 2008)

There are so many integration techniques to cover in the literature, but you will find a few of the well-known of them for addressing the problems with integrals. The most common approaches of integration include; *Integrating Functions Using Long Division*, *the Power Rule for Integration*, *Integration by Substitution*, *Trigonometric Substitution*, *Integration by Parts*, *Integration by Partial Fractions*, *Integrals of Inverse Functions*, *Weierstrass Substitution*, and *Feynman's Integration Method*. These are essential to know, but they will not always be helpful when dealing with improper integrals.

Among all the approaches that have been mentioned above, we found that the most practical approach to calculate the actual values of the so-called "non-elementary integrals" is provided by "Feynman's method" (Kotikov, 1991). We refer the reader for additional information on the integration methods listed above to some appropriate resources (Ashlock, 2019) and (Shaxobiddinova et al, 2022).

In this paper, Feynman's method will be discussed in details along with some illustrated examples involving improper integral formulae.

2 Basic Definitions

This section presents the basic definitions that have been used in this paper: we have introduced Leibniz Rule (LR) and the Initial Value Problem (IVP) , and presented a brief explanation of the ,

Definition 1. (Leibniz Rule) : Let a and b be constants. If a function f and its partial derivative $\frac{\partial f}{\partial x}$ are continuous on a rectangle $R = \{(t, x): a \leq t \leq b, c \leq x \leq d\}$, then

$$I(x) = \int_a^b f(t, x)\, dt$$

has the derivative

$$\frac{dI(x)}{dx} = \int_a^b \frac{\partial f(t, x)}{\partial x}\, dt.$$

The Leibniz integral rule is another name for the differentiation under the integral sign rule. A formula for solving a definite integral with functions of the differential variable acting as long as its limits exist. This rule makes it possible to compute a differentiation of integral without integrating the equation.

Definition 2. (Initial Value Problem) : An Initial Value Problem, also known as an IVP, is a problem in which we need to solve a differential equation $\frac{dy}{dx} = f(x, y)$ with a solution that satisfies $y(x_0) = y_0$.

Feynman's method

Feynman's method is often applied to integrals with a single variable. The technique of Feynman itself is not easy to put into words. However, the idea is to artificially introduce a second variable and then build a differential equation in terms of derivatives with respect to the new variable. By solving this differential equation we often can compute the original integral.

The following steps provide at least a sketch of what we mean when evaluating the definite integral of a function; $I = \int_a^b f(x)dx$ using *Feynman's method*:

1. Consider the integral I as a function of p; where p is an arbitrary parameter,

$$I(p) = \int_a^b f(x, p)\ dx$$

2. Compute the integral for some particular convenient value of $p = p_0$, such that $I(p_0) = I_0$.
3. Differentiate the integral with respect to p.
4. Compute the definite integral with respect to x.
5. Integrate indefinitely with respect to p.
6. Use the fact that $I(p_0) = I_0$ to compute the value of the constant of integration.

Following the previous steps, the problem will be changed from computing an integral to solving a straightforward differential equation. We warm up with a relatively simple examples.

3 Examples

Example 3.1.

Consider the function $f(x) = \frac{\tan^{-1} ax - \tan^{-1} bx}{x}$, where a, b are any constants such that $\frac{a}{b} > 0$. Let us compute the following Integral:

$$I = \int_0^\infty f(x)\,dx = \int_0^\infty \frac{\tan^{-1} ax\ - \tan^{-1} bx}{x} dx \tag{1}$$

We first Introduce I as a function of a parameter p:

$$I(p) = \int_0^\infty \frac{\tan^{-1} px\ - \tan^{-1} bx}{x} dx \tag{2}$$

Such that $I(a) = I$, and $I(b) = 0$.

Now, differentiate both sides of Eq. (2) with respect to p;

$$\frac{d}{dp} I(p) = \int_0^\infty \frac{\partial}{\partial p}\left(\frac{\tan^{-1} px\ - \tan^{-1} bx}{x}\right) dx \tag{3}$$

$$\frac{d}{dp} I(p) = \int_0^\infty \frac{1}{x}\frac{\partial}{\partial p}(\tan^{-1} px\ ) dx \tag{4}$$

$$\frac{d}{dp} I(p) = \int_0^\infty \frac{1}{x}\left(\frac{x}{1 + (px)^2}\right) dx \tag{5}$$

We can now calculate the integration by taking $u = px$, and $du = p\,dx$.

$$\frac{d}{dp} I(p) = \frac{1}{p}\int_0^\infty \left(\frac{1}{1 + u^2}\right) du \tag{6}$$

Then,

$$\frac{d}{dp} I(p) = \frac{1}{p} u \ |_0^\infty \tag{7}$$

Let us now return u back to its original value px;

$$\frac{d}{dp} I(p) = \frac{1}{p} px \ |_0^\infty \tag{8}$$

By substituting, we have;

$$\frac{d}{dp} I(p) = \frac{\pi}{2p}; \qquad I(b) = 0 \tag{9}$$

By solving the initial value problem in Eq. (9), we get;

$$I(p) = \frac{\pi}{2} \ln p \ + c \qquad (10$$

We can find the value of c, by replacing p with b in Eq. (10);

$$I(b) = \frac{\pi}{2} ln \ b \ + c. \qquad (11$$

Since $I(b) = 0$, then

$$0 = \frac{\pi}{2} ln \ b \ + c, \qquad (12$$

So, we get

$$c = - \frac{\pi}{2} ln \ b \ . \qquad (13$$

By subtitling the value of c in Eq. (10), we get:

$$I(p) = \frac{\pi}{2} ln \ p \ - \frac{\pi}{2} b \qquad (14$$

$$I(p) = \frac{\pi}{2} ln \ \left(\frac{p}{b}\right) \qquad (15$$

Therefore,

$$I(a) = I = \int_0^{\infty} \frac{\tan^{-1} ax - \tan^{-1} bx}{x} dx = \frac{\pi}{2} ln\left(\frac{a}{b}\right) \quad (16$$

We should take in our consideration that $\frac{a}{b} > 0$. If we put $a = 2$ and $b = 8$, we get $f(x) = \frac{\tan^{-1} 2x - \tan^{-1} 8x}{x}$. Figure 1. shows the graph of the function $f(x)$ and its antiderivative $F(x) = \int_0^x f(t)dt$. Notice that as $x \to \infty$, $F(x) \to I$. From Eq. (16) we have $I = \frac{\pi}{2} ln\left(\frac{2}{8}\right) = \frac{\pi}{2} ln\left(\frac{1}{4}\right) \cong -2.17758609$.

Example 3.2.

Let us take the function $(x) = \frac{x^a - 1}{\ln(x)}$; provided that $a > -1$. Since the process of finding the definite Integral of this function is not easy, we will follow the Feynman's method:

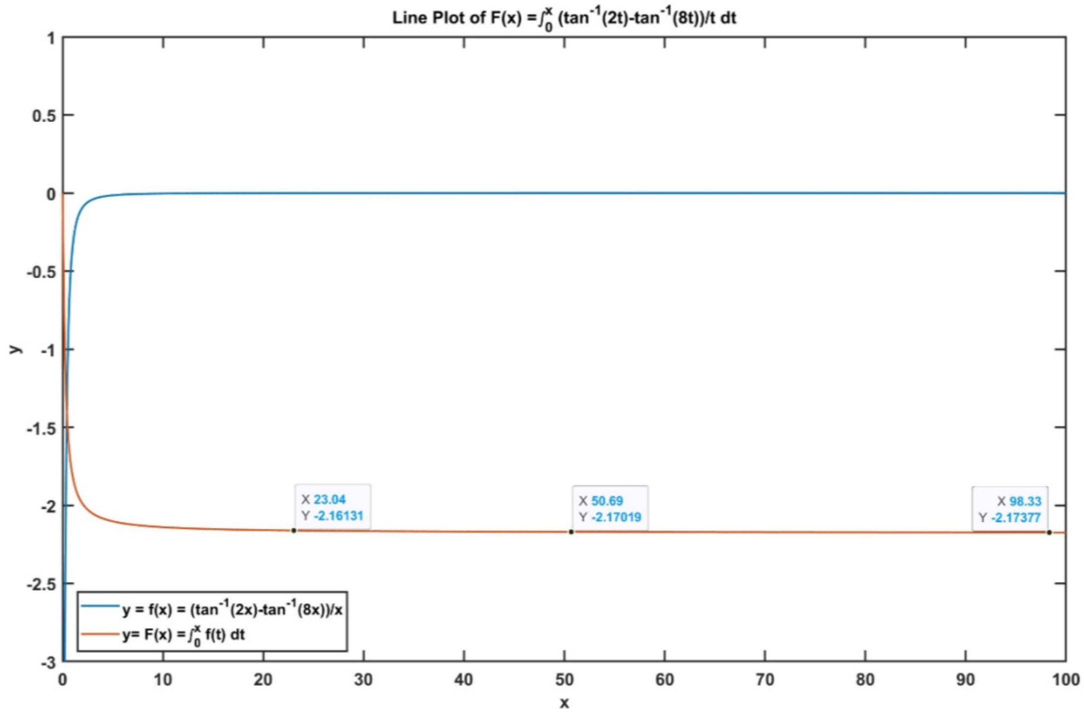

Fig. 1. The graph of $f(x) = \frac{\tan^{-1} 2x - \tan^{-1} 8x}{x}$ and its antiderivative within the domain $0 \leq x \leq 100$.

$$I = \int_0^1 f(x)dx = \int_0^1 \frac{x^a - 1}{\ln(x)} dx; \quad a > -1 \tag{17}$$

Again, for computing the integration in Eq. (17), we Introduce I as a function in parameter p as follows:

$$I(p) = \int_0^1 \frac{x^p - 1}{ln\,(x)} dx; \tag{18}$$

Such that $I(a) = I$, and $I(0) = 0$.

Now, differentiate both sides of Eq. (18) with respect to p

$$\frac{d}{dp} I(p) = \int_0^1 \frac{1}{ln\,(x)} \frac{\partial}{\partial p} (x^p - 1) dx \tag{19}$$

$$\frac{d}{dp} I(p) = \int_0^1 \frac{1}{ln\,(x)} (x^p . (\,ln\,(x)\,) dx \tag{20}$$

$$\frac{d}{dp} I(p) = \int_0^1 x^p dx \tag{21}$$

We can now calculate the integration by taking $u = px$, and $du = p\,dx$.

$$\frac{d}{dp} I(p) = \frac{x^{p+1}}{p+1} |_0^1 \tag{22}$$

Then,

$$\frac{d}{dp} I(p) = \frac{1}{p+1}; \quad I(0) = 0. \tag{23}$$

Taking the integral of both sides, we get;

$$I(p) = ln\,|p + 1| + c \tag{24}$$

To find the value of c, we replace p with 0 as follows:

$$I(0) = \; ln\,|0 + 1| + c\,. \tag{25}$$

Since $I(0) = 0$, then

$$0 = \ ln\,|1| + c\ , \tag{26}$$

and we get

$$c = 0 \tag{27}$$

By subtitling the value of c in Equation 24, we have:

$$I(p) = ln\,|p+1| \tag{28}$$

Therefore, our Integration I became

$$I = I(a) = ln\,|a+1| \tag{29}$$

When $a = 1$, we have $(x) = \frac{x-1}{ln\,x}$. The definite integral $I = \int_0^1 f(x)dx = \int_0^1 \frac{x-1}{ln\,x}dx = ln\,(2) \cong 0.69314718$. The graph of the function $f(x) = \frac{x-1}{ln\,x}$ and its antiderivatives $F(x) = \int_0^x f(t)\,dt$ is shown in Figure 2.

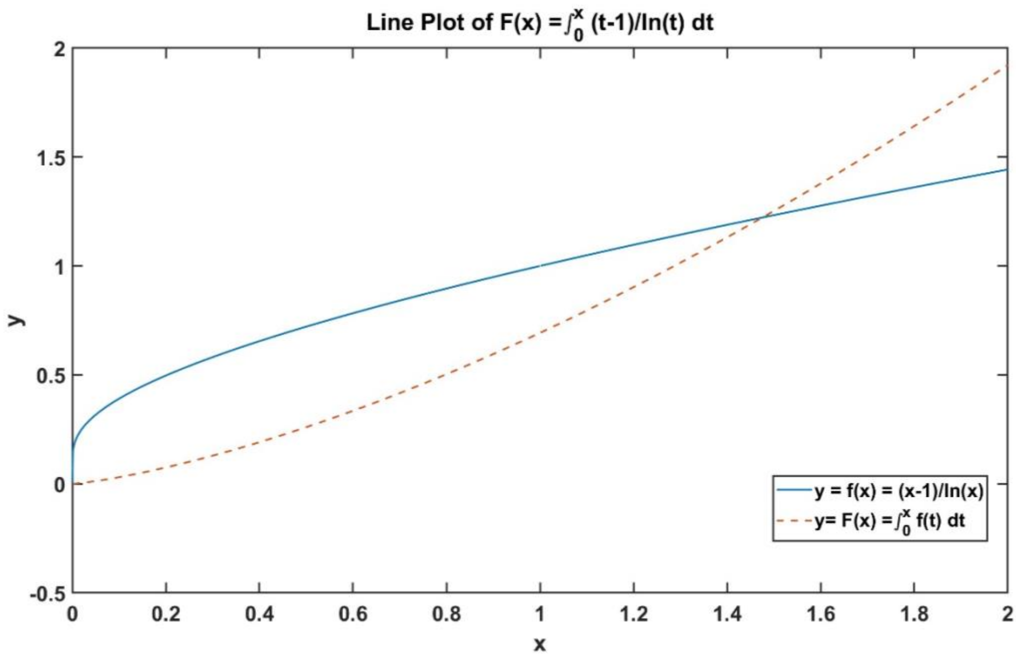

Fig. 2. The graph of $f(x) = \frac{x-1}{\ln x}$ and its antiderivative within the domain $0 \le x \le 2$.

Example 3.3.

Here, we consider the integral

$$I = \int_0^{\frac{\pi}{2}} x\, cot\,(x)\, dx.$$

First, and before choosing a proper parameter p, we need to make some manipulation with the function $xcot\,(x)$ as follows:

Let $x\, cot\,(x) \; = \frac{x}{tan\,(x)}$, so our integration becomes;

$$I = \int_0^{\frac{\pi}{2}} \frac{x}{tan\,(x)} dx \tag{30}$$

Now we can introduce $I(p)$ and rewrite the integral as

$$I(p) = \int_0^{\frac{\pi}{2}} \frac{x}{tan\,(px)} dx \tag{31}$$

where, $I(1) = I$, and $I(0) = 0$.

Differentiating both sides of Eq. (31) with respect to p to have

$$\frac{d}{dp} I(p) = \int_0^{\frac{\pi}{2}} \left(\frac{1}{tan\,(x)} \frac{\partial}{\partial p} x)\;\right) dx \tag{32}$$

$$\frac{d}{dp} I(p) = \int_0^{\frac{\pi}{2}} \left(\frac{1}{tan\,(x)} \left[\frac{\tan x}{1+p^2 x}\right]\right) dx$$
$$= \int_0^{\frac{\pi}{2}} \left(\frac{1}{1+p^2 x}\right) dx. \tag{33}$$

Let $u = tan \;\; x$, $du = x \;\; dx$ (performing *Integration by substitution* method), we get

$$\frac{d}{dp} I(p) = \int_0^{\infty} \frac{1}{1+p^2u^2} \; \frac{du}{1+u^2}. \tag{34}$$

We can now calculate the integration by using the partial *fraction* method. So the integral in the right hand side becomes

$$\frac{d}{dp} I(p) = \frac{\pi}{2}\left(\frac{1}{p+1}\right), \qquad I(0) = 0. \tag{35}$$

Taking the integral of both sides, we get

$$I(p) = \frac{\pi}{2} ln\, |p+1| + c \tag{36}$$

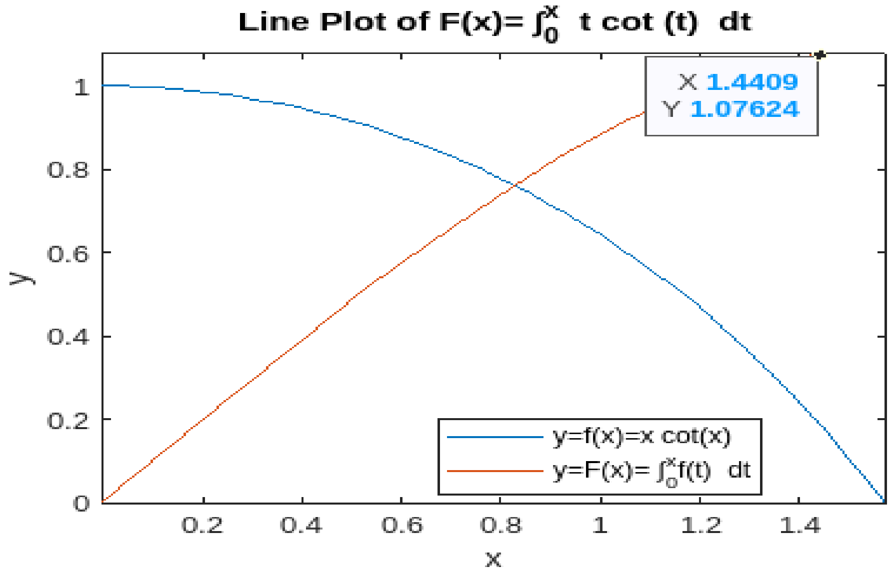

Fig. 3. The graph of $f(x) = x\, cot\,(x)$ and its antiderivative within the domain $0 \leq x \leq 1.6$.

To find the value of c, we replace p with 0 as follows:

$$I(0) = \tfrac{\pi}{2}(ln\, |0+1|\,) + c. \tag{37}$$

But $I(0) = 0$, then

$$0 = \frac{\pi}{2} ln\,(1)\; + c, \tag{38}$$

and we get

$$c = 0 \tag{39}$$

By subtitling the value of c in Equation (36), we have

$$I(p) = \frac{\pi}{2} ln\ (p+1) \tag{40}$$

Therefore,

$$I = I(1) = \frac{\pi}{2} ln\ (2) \cong 1.08879\,. \tag{41}$$

is shown in

Figure 3. shows the graph of the function $f(x) = x\cot x$ along with its antiderivatives $F(x) = \int_0^x f(t)\,dt$.

Table1. Using Feynman's method for calculating the definite integrals of some non-elementary functions

The definite Integral	**The assumption for parameter *p***	**The initial conditions**	**The Solution as a function of *p***
$\boldsymbol{I_1 = \int_0^\infty \frac{ln\ (1+4x^2)}{x^2} d}$	$I_1(p) = \int_0^\infty \frac{ln\ (1+px^2)}{x^2} dx$	$I_1(4) = I_1$, $I_1(0) = 0$	$I_1(p) = \pi\sqrt{p}$
$\boldsymbol{I_2 = \int_0^\infty \frac{sin\ (x^2)}{x^2} dx}$	$I_2(p) = \int_0^\infty \frac{sin\ (px^2)}{x^2} dx$	$I_2(1) = I_2$, $I_2(0) = 0$	$I_2(p) = \frac{\sqrt{2\pi p}}{2}$
$\boldsymbol{I_3 = \int_0^\infty \frac{sin^2(x)}{x^2} dx}$	$I_3(p) = \int_0^\infty \frac{sin^2(px)}{x^2} dx$	$I_3(1) = I_2$, $I_3(0) = 0$	$I_3(p) = \frac{\pi p^2}{2}$
$\boldsymbol{I_4 = \int_0^\infty \frac{e^{-2x}\ sin\ (x)}{x} c}$	$I_4(p) = \int_0^\infty \frac{e^{-px} sin(x)}{x} dx$	$I_4(2) = I_4$, $I_4(\infty) = 0$	$I_4(p) = p$
$\boldsymbol{I_5 = \int_0^\infty \frac{cos(x)}{1+x^4} dx}$	$I_5(p) = \int_0^\infty \frac{cos\ (px)}{1+x^4} dx$	$I_5(1) = I_5$, $I_5(0) = \frac{\pi}{2}$	$I_5(p) = \frac{\pi\ \ sin\ \left(\frac{p}{2}\right) cos}{2\sqrt{2}\, e^{\frac{\sqrt{2}}{2}}}$

$\boldsymbol{I_6 = \int_0^{\frac{\pi}{2}} x \ \ cot\, x\, dx}$	$I_6(p) = \int_0^{\frac{\pi}{2}} \frac{(p\, tan\ \ x\,)}{tan\ \ x} dx$	$I_6(1) = I_7,$ $I_7(0) = 0$	$I_6(p) = \frac{\pi}{2}\ \ ln\,(p + 1)$
$\boldsymbol{I_7 = \int_0^{\pi} \frac{ln\,(1 + cos\, x}{cos\, x}}$	$I_7(p) = \int_0^{\pi} \frac{ln\,(1 + cos\ \ p\, cos}{cos\ \ x}$	$I_7(0) = I_7,$ $I_7\left(\frac{\pi}{2}\right) = 0$	$I_7(p) = \pi(\frac{\pi}{2} - p)$
$\boldsymbol{I_8 = \int_0^{\infty} e^{-x^2} cos\, x\, dx}$	$I_8(p) = \int_0^{\infty} e^{-x^2}$ $cos\ \ px\, dx$	$I_8(1) = I_8,$ $I_8\left(\frac{\pi}{2}\right) = 0$	$I_8(p) = \frac{\sqrt{\pi}}{2} e^{\frac{-p^2}{4}}$
$\boldsymbol{I_9 = \int_0^{\infty} \frac{sin\,(x^3)}{x} dx}$	$I_9(p) = \int_0^{\infty} e^{-px^3} \frac{sin\,(x^3)}{x} dx$	$I_9(0) = I_9$ $, I_9(\infty) = 0$	$I_9(p) = \frac{-1}{3} p + \frac{\pi}{6}$

In Table 1. We present a few examples of improper integrals that have been solved using the Feynman's method. It could be noticed that the Feynman method was much simpler to use in the time when it was difficult to find real solutions for some integrals, even with computer software.

4 Conclusions and Remarks

The Feynman method has been used a lot when dealing with difficult integrations of physical and mathematical problems. It is known that the ability of generalization is one of the most important skills for solving mathematical problems, and this was accomplished using this method; since we can get the solution as formula or a function of a variable p, and this allows us to produce so many solutions based on the value of p.

The main idea behind " Feynman's method " is to solve a simple differential equation instead of computing an integral. This method can be used to test the convergence and divergence series. Getting the actual value of the integration is better than getting the approximate value; for example, Taylor series can be used to solve the definite integral, but unlike Feynman, it does not represent the actual value.

The only drawback to this approach is that when you begin to solve the integral, it is hard to predict the order of the differential equation. There is no clear algorithm to construct a function of a parameter p; each problem has unique assumptions about p. Therefore, designing a program to create $I(p)$ is difficult. The illustrated examples in this paper demonstrated how well Feynman's method works to solve challenging integrations. The majority of the examples presented in Table 1. shows the results of using Feynman's method for formulae involve improper integral.

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